Jordan Journal of Mathematics and Statistics (JJMS), 16(3), 2023, pp 563 - 584
DOI: https://doi.org/10.47013/16.3.12

FRACTIONAL SIMPSON LIKE TYPE INEQUALITIES FOR DIFFERENTIABLE s-CONVEX FUNCTIONS

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Abstract. Convexity inequalities are very important for fractional calculus and its efficiency in many applied sciences. This field has become increasingly popular and represents a powerful tool for estimating errors of quadrature formulas. In this paper, we seek to develop new four-point Simpson-type inequalities involving Riemenn-Liouville integral operators. To do this, we first propose a new integral identity. By using this identity we establish some new fractional Simpson like type inequalities for functions whose first derivatives are s-convex in the second sense. Some particular cases are also discussed. We provid at the end some applications to special means to demonstrate the effectiveness of our results.

1. Introduction

Definition 1.1. [15] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Convexity theory plays an important and central role in many fields such as economics, finance, optimization, and game theory. Due to its rapid development, several

2010 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.
Key words and phrases. 3/8-Simpson inequality, Riemann-Liouville integral operators, s-convex functions, Hölder inequality.

Received: July 24, 2022 Accepted: Dec. 14, 2022 .

researchers have introduced new classes of convex functions, among these classes we recall that introduced by Breckner called class of s-convex functions.

Definition 1.2. [2] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \ \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

It is clear that the convexity has a close relationship in the development of the theory of inequalities, of which it plays an important role in numerical analysis more precisely in the estimation of the errors of the quadrature rules see [1, 3, 4, 6–12, 16, 17, 20–22] and references therein.

Two of the most famous quadratures are Simpson's formulas. The first formula also called 1/8-Simpson inequality (see [5]), is as follows:

$$\left| \tfrac{1}{6} \left(f\left(a\right) + 4f\left(\tfrac{a+b}{2}\right) + f\left(b\right) \right) - \tfrac{1}{b-a} \int\limits_a^b f\left(u\right) du \right| \leq \tfrac{(b-a)^4}{2880} \left\| f^{(4)} \right\|_\infty . \tag{1.1}$$

Simpson's second formula also called 3/8-Simpson inequality (see [18]) is formulated as follows

$$\left| \tfrac{1}{8} \left(f\left(a\right) + 3f\left(\tfrac{2a+b}{3}\right) + 3f\left(\tfrac{a+2b}{3}\right) + f\left(b\right) \right) - \tfrac{1}{b-a} \int\limits_a^b f\left(u\right) du \right| \leq \tfrac{(b-a)^4}{6480} \left\| f^{(4)} \right\|_\infty , \tag{1.2}$$

where f is four-times continuously differentiable function on (a, b), and $\left\| f^{(4)} \right\|_\infty = \sup\limits_{x \in (a,b)} \left| f^{(4)}\left(x\right) \right|$.

In [19], Sarikaya et al., gave the following Simpson for s-convex first defivatives

$$\left| \tfrac{1}{6} \left(f\left(a\right) + 4f\left(\tfrac{a+b}{2}\right) + f\left(b\right) \right) - \tfrac{1}{b-a} \int\limits_a^b f\left(u\right) du \right|$$

$$\leq \frac{b-a}{12}\left(\frac{1+2^{1+p}}{3(1+p)}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

and

$$\left|\frac{1}{6}\left(f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{2}\left(\frac{5}{36}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}|f'(a)|^q+\frac{2\times 5^{s+2}+6^{s+1}(s-4)-3^{s+1}(2s+7)}{3\times 6^{s+1}(s+1)(s+2)}|f'(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{2\times 5^{s+2}+6^{s+1}(s-4)-3^{s+1}(2s+7)}{3\times 6^{s+1}(s+1)(s+2)}|f'(a)|^q+\frac{(2s+1)3^{s+1}+2}{3\times 6^{s+1}(s+1)(s+2)}|f'(b)|^q\right)^{\frac{1}{q}}\right).$$

Recently, Mahmoudi and Meftah [14], established the following 3/8-Simpson type inequalities for s-convex derivatives

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{9(s+1)(s+2)}\left(\left(\frac{3s-2}{8}+2\left(\frac{5}{8}\right)^{s+2}\right)(|f'(a)|+|f'(b)|)\right.$$
$$\left.+\left(\frac{9s+2}{8}+\left(\frac{1}{2}\right)^{s+1}+2\left(\frac{3}{8}\right)^{s+2}\right)\left(\left|f'\left(\frac{2a+b}{3}\right)\right|+\left|f'\left(\frac{a+2b}{3}\right)\right|\right)\right),$$

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{18(p+1)^{p+1}}\left(\left(\frac{3^{p+1}+5^{p+1}}{2^{2p+3}}\right)^{\frac{1}{p}}\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+5^{p+1}}{2^{2p+3}}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

and

$$\left|\frac{1}{8}\left(f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq \frac{b-a}{9((s+1)(s+2))^{\frac{1}{q}}}\left(\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\frac{3s-2}{8}+2\left(\frac{5}{8}\right)^{s+2}\right)\left|f'(a)\right|^q\right.\right.$$
$$\left.+\left(\frac{5s+2}{8}+2\left(\frac{3}{8}\right)^{s+2}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\frac{1}{4}\left(2s+\left(\frac{1}{2}\right)^{s-1}\right)^{\frac{1}{q}}\left(\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\frac{5s+2}{8}+2\left(\frac{3}{8}\right)^{s+2}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right.$$
$$\left.\left.+\left(\frac{3s-2}{8}+2\left(\frac{5}{8}\right)^{s+2}\right)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right).$$

In the last decades fractional calculus has attracted the attention of many researchers due to its has wide applications in pure and applied mathematics, especially the Riemann-Liouville operator several fractional analogues and new integral inequalities have been established. The Riemann-Liouville integral is defined as follows:

Definition 1.3. [13] Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integrals $I_{a^+}^{\alpha}f$ and $I_{b^-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$
$$I_{b^-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x$$

respectively, where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$, is the Gamma function and $I_{a^+}^0 f(x) = I_{b^-}^0 f(x) = f(x)$.

In [10], Kamouche et al., proposed the following fractional Simpson like type inequalities for differentiable s-convex functions.

$$\left|\frac{1}{8}\left(f(a)+6f\left(\frac{a+b}{2}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)+I_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)\right)\right|$$
$$\leq \frac{b-a}{4}\left(\Theta_{s,\alpha}\left|f'(a)\right|+\frac{\alpha+2^{\frac{2s+2}{\alpha}-1}(3s+3-\alpha)}{2^{\frac{2s+2}{\alpha}}(s+1)(\alpha+s+1)}\left|f'\left(\frac{a+b}{2}\right)\right|+\Theta_{s,\alpha}\left|f'(b)\right|\right)$$

and

$$\begin{aligned}
&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)+I^{\alpha}_{\left(\frac{a+b}{2}\right)^{+}}f(b)\right)\right| \\
\leq\ & \tfrac{b-a}{4}\left(\tfrac{1}{4^{p+\frac{1}{\alpha}}\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\tfrac{3^{p+1}}{4^{p+1}\alpha(p+1)}\cdot{}_{2}F_{1}\left(1-\tfrac{1}{\alpha},1,p+2;\tfrac{3}{4}\right)\right)^{\frac{1}{p}} \\
&\times\left(\left(\tfrac{|f'(a)|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+|f'(b)|^{q}}{s+1}\right)^{\frac{1}{q}}\right),
\end{aligned}$$

where

$$\begin{aligned}
\Theta_{s,\alpha} &= \tfrac{1}{4(s+1)}\left(1-2\left(1-\tfrac{1}{\sqrt[\alpha]{4}}\right)^{s+1}\right) \\
&\quad +B_{1-\frac{1}{\sqrt[\alpha]{4}}}\left(s+1,\alpha+1\right)-B_{\frac{1}{\sqrt[\alpha]{4}}}\left(\alpha+1,s+1\right).
\end{aligned}$$

Motivated by the above results, in this paper, we first propose a new identity as partial result. On the basis of this identity, we establish new Simpson type inequalities for functions whose first derivatives are s-convex in the second direction involving the integral operators of Riemann-Liouville. Some particular cases are derived. Applications to special means are provided to demonstrate the efficacy of our results.

2. Main results

We first recall some special functions which will be used in the sequel.

Definition 2.1. [13] For any complex numbers x, y such that $\operatorname{Re}(x) > 0$ and $\operatorname{Re}(y) > 0$. The beta function is defined by

$$B(x,y)=\int_{0}^{1}t^{x-1}(1-t)^{y-1}\,dt=\tfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

where $\Gamma(.)$ is the Euler gamma function. The incomplete beta function.

Definition 2.2. [13] For any complex numbers and nonpositive integers x, y such that $\mathrm{Re}\,(x) > 0$ and $\mathrm{Re}\,(y) > 0$. The incomplete beta function is given by

$$B_a\left(x,y\right)=\int\limits_0^a t^{x-1}\left(1-t\right)^{y-1}dt,\quad a<1.$$

Definition 2.3. [13] The hypergeometric function is defined for $\mathrm{Re}c > \mathrm{Re}b > 0$ and $|z| < 1$, as follows

$${}_2F_1\left(a,b,c;z\right)=\tfrac{1}{B(b,c-b)}\int\limits_0^1 t^{b-1}\left(1-t\right)^{c-b-1}\left(1-zt\right)^{-a}dt,$$

where $c > b > 0, |z| < 1$ and $B\left(.,.\right)$ is the beta function.

We start with the following identity which is crucial to establish our main results

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L^1\left[a,b\right]$, then the following equality holds*

$$\begin{aligned}
&\tfrac{1}{6}\left(f\left(a\right)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f\left(b\right)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_\alpha\\
=\;&\tfrac{b-a}{36}\Bigg(\int\limits_0^1\left(4t^\alpha-2\right)f'\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)dt\\
&-\int\limits_0^1\left(1-t\right)^\alpha f'\left(\left(1-t\right)\tfrac{2a+b}{3}+t\tfrac{a+b}{2}\right)dt\\
&+\int\limits_0^1 t^\alpha f'\left(\left(1-t\right)\tfrac{a+b}{2}+t\tfrac{a+2b}{3}\right)dt\\
&-\int\limits_0^1\left(4\left(1-t\right)^\alpha-2\right)f'\left(\left(1-t\right)\tfrac{a+2b}{3}+tb\right)dt\Bigg),
\end{aligned}$$

where

$$\mathcal{S}_\alpha=I^\alpha_{\left(\frac{2a+b}{3}\right)^-}f(a)+2^{\alpha-1}I^\alpha_{\left(\frac{a+2b}{3}\right)^-}f\left(\tfrac{a+b}{2}\right)+2^{\alpha-1}I^\alpha_{\left(\frac{2a+b}{3}\right)^+}f\left(\tfrac{a+b}{2}\right)+I^\alpha_{\left(\frac{a+2b}{3}\right)^+}f(b). \tag{2.1}$$

Proof. Let

$$I = I_1 - I_2 + I_3 - I_4, \tag{2.2}$$

where

$$\begin{aligned} I_1 &= \int_0^1 (4t^\alpha - 2) f'\left((1-t)a + t\tfrac{2a+b}{3}\right) dt, \\ I_2 &= \int_0^1 (1-t)^\alpha f'\left((1-t)\tfrac{2a+b}{3} + t\tfrac{a+b}{2}\right) dt, \\ I_3 &= \int_0^1 t^\alpha f'\left((1-t)\tfrac{a+b}{2} + t\tfrac{a+2b}{3}\right) dt \end{aligned}$$

and

$$I_4 = \int_0^1 (4(1-t)^\alpha - 2) f'\left((1-t)\tfrac{a+2b}{3} + tb\right) dt.$$

Integrating by parts I_1, we obtain

$$\begin{aligned} I_1 &= \tfrac{3}{b-a}(4t^\alpha - 2) f\left((1-t)a + t\tfrac{2a+b}{3}\right)\Big|_0^1 \\ &\quad -\alpha\left(\tfrac{12}{b-a}\right)\int_0^1 t^{\alpha-1} f\left((1-t)a + t\tfrac{2a+b}{3}\right) dt \\ &= \tfrac{6}{b-a} f\left(\tfrac{2a+b}{3}\right) + \tfrac{6}{b-a} f(a) - 4\alpha\left(\tfrac{3}{b-a}\right)^{\alpha+1} \int_a^{\frac{2a+b}{3}} (u-a)^{\alpha-1} f(u)\, du \\ &= \tfrac{6}{b-a} f\left(\tfrac{2a+b}{3}\right) + \tfrac{6}{b-a} f(a) - \tfrac{4\times 3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^\alpha_{\left(\frac{2a+b}{3}\right)^-} f(a). \end{aligned} \tag{2.3}$$

Similarly, we obtain

$$I_2 = -\tfrac{6}{b-a} f\left(\tfrac{2a+b}{3}\right) + \tfrac{6^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^\alpha_{\left(\frac{2a+b}{3}\right)^+} f\left(\tfrac{a+b}{2}\right), \tag{2.4}$$

$$I_3 = \tfrac{6}{b-a} f\left(\tfrac{a+2b}{3}\right) - \tfrac{6^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^\alpha_{\left(\frac{a+2b}{3}\right)^-} f\left(\tfrac{a+b}{2}\right) \tag{2.5}$$

and

$$I_4=-\tfrac{6}{b-a}f\left(b\right)-\tfrac{6}{b-a}f\left(\tfrac{a+2b}{3}\right)+\tfrac{4\times(3)^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}}I^{\alpha}_{\left(\frac{a+2b}{3}\right)^{+}}f(b). \tag{2.6}$$

Using (2.3)-(2.6) in (2.2), and then multiplying the resulting equality by $\frac{b-a}{36}$, we get the desired result. □

Theorem 2.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If $|f'|$ is s-convex in the second sense for some fixed $s\in(0,1]$, then we have*

$$\begin{aligned}
&\left|\tfrac{1}{6}\left(f\left(a\right)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f\left(b\right)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right|\\
\leq\ &\tfrac{b-a}{36}\left(\left(\tfrac{2}{s+1}\left(1-2\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)+\mathcal{L}\left(s+1,\alpha+1\right)\right)\left(\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right)\right.\\
&+\left(\tfrac{3(s+1)-2\alpha}{(s+1)(\alpha+s+1)}+\tfrac{\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-2\alpha}{\alpha}}\right)\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|+\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)\\
&\left.+\,2B\left(s+1,\alpha+1\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|\right),
\end{aligned}$$

where

$$\mathcal{L}\left(x,y\right)=4B_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(x,y\right)-4B_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(y,x\right), \tag{2.7}$$

$\mathcal{S}_{\alpha}$ is defined by (2.1) and $B\left(.,.\right)$ is the beta function.

Proof. From Lemma 2.1, properties of modulus, and s-convexity in the second sens of $|f'|$, we have

$$\begin{aligned}
&\left|\tfrac{1}{6}\left(f\left(a\right)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f\left(b\right)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right|\\
\leq\ &\tfrac{b-a}{36}\left(\int_0^1\left|4t^{\alpha}-2\right|\left|f'\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right|dt\right.\\
&+\int_0^1\left(1-t\right)^{\alpha}\left|f'\left(\left(1-t\right)\tfrac{2a+b}{3}+t\tfrac{a+b}{2}\right)\right|dt
\end{aligned}$$

$$
\begin{aligned}
& +\int_0^1 t^{\alpha}\left|f'\left((1-t)\tfrac{a+b}{2}+t\tfrac{a+2b}{3}\right)\right|dt \\
& \left.+\int_0^1 \left|4(1-t)^{\alpha}-2\right|\left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)\right|dt\right) \\
\leq\ & \tfrac{b-a}{36}\left(\int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(2-4t^{\alpha}\right)\left((1-t)^s\left|f'(a)\right|+t^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|\right)dt\right. \\
& +\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1\left(4t^{\alpha}-2\right)\left((1-t)^s\left|f'(a)\right|+t^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|\right)dt \\
& +\int_0^1 (1-t)^{\alpha}\left((1-t)^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)dt \\
& +\int_0^1 t^{\alpha}\left((1-t)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|+t^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)dt \\
& +\int_0^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4(1-t)^{\alpha}-2\right)\left((1-t)^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|+t^s\left|f'(b)\right|\right)dt \\
& \left.+\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1\left(2-4(1-t)^{\alpha}\right)\left((1-t)^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|+t^s\left|f'(b)\right|\right)dt\right) \\
=\ & \tfrac{b-a}{36}\left(\left|f'(a)\right|\left(\int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(2-4t^{\alpha}\right)(1-t)^s\,dt+\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1\left(4t^{\alpha}-2\right)(1-t)^s\,dt\right)\right.
\end{aligned}
$$

$$+\left|f'\left(\tfrac{2a+b}{3}\right)\right|\left(\int_{0}^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(2-4t^{\alpha}\right)t^{s}dt+\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(4t^{\alpha}-2\right)t^{s}dt+\int_{0}^{1}\left(1-t\right)^{\alpha+s}dt\right)$$

$$+\left|f'\left(\tfrac{a+b}{2}\right)\right|\left(\int_{0}^{1}\left(1-t\right)^{\alpha}t^{s}dt+\int_{0}^{1}t^{\alpha}\left(1-t\right)^{s}dt\right)$$

$$+\left|f'\left(\tfrac{a+2b}{3}\right)\right|\left(\int_{0}^{1}t^{\alpha+s}dt+\int_{0}^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4\left(1-t\right)^{\alpha}-2\right)\left(1-t\right)^{s}dt\right.$$

$$\left.+\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(2-4\left(1-t\right)^{\alpha}\right)\left(1-t\right)^{s}dt\right)$$

$$\left.+\left|f'\left(b\right)\right|\left(\int_{0}^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4\left(1-t\right)^{\alpha}-2\right)t^{s}dt+\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(2-4\left(1-t\right)^{\alpha}\right)t^{s}dt\right)\right)$$

$$=\frac{b-a}{36}\left(\left(\frac{2}{s+1}\left(1-2\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)+\mathcal{L}\left(s+1,\alpha+1\right)\right)\left(\left|f'\left(a\right)\right|+\left|f'\left(b\right)\right|\right)\right.$$

$$+\left(\frac{3(s+1)-2\alpha}{(s+1)(\alpha+s+1)}+\frac{\alpha}{(s+1)(\alpha+s+1)}\left(\frac{1}{2}\right)^{\frac{s+1-2\alpha}{\alpha}}\right)\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|+\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)$$

$$\left.+2B\left(s+1,\alpha+1\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|\right),$$

where we have used the fact that

$$\int_{0}^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(2-4t^{\alpha}\right)\left(1-t\right)^{s}dt=\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(2-4\left(1-t\right)^{\alpha}\right)t^{s}dt \tag{2.8}$$

$$= \tfrac{2}{s+1}\left(1-\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)-4B_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(\alpha+1,s+1\right),$$

$$\int\limits_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(4t^{\alpha}-2\right)\left(1-t\right)^{s}dt \tag{2.9}$$

$$= \int\limits_{0}^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4\left(1-t\right)^{\alpha}-2\right)t^{s}dt$$

$$= 4B_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(s+1,\alpha+1\right)-\tfrac{2}{s+1}\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1},$$

$$\int\limits_{0}^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(2-4t^{\alpha}\right)t^{s}dt \tag{2.10}$$

$$= \int\limits_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(2-4\left(1-t\right)^{\alpha}\right)\left(1-t\right)^{s}dt=\tfrac{\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-\alpha}{\alpha}}$$

$$\int\limits_{0}^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4\left(1-t\right)^{\alpha}-2\right)\left(1-t\right)^{s}dt \tag{2.11}$$

$$= \int\limits_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(4t^{\alpha}-2\right)t^{s}dt=\tfrac{2(s+1)-2\alpha}{(s+1)(\alpha+s+1)}+\tfrac{\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-\alpha}{\alpha}},$$

$$\int\limits_{0}^{1}\left(1-t\right)^{\alpha+s}dt=\int\limits_{0}^{1}t^{\alpha+s}dt=\tfrac{1}{\alpha+s+1} \tag{2.12}$$

and

$$\int\limits_{0}^{1}\left(1-t\right)^{\alpha}t^{s}dt=B\left(s+1,\alpha+1\right). \tag{2.13}$$

The proof is completed. □

Corollary 2.1. *In Theorem 2.1, if we take $s = 1$, then we get*

$$\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right|$$

$$\leq \tfrac{b-a}{36}\left(\left(\tfrac{4-(\alpha+1)(\alpha+2)}{(\alpha+1)(\alpha+2)}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}-\tfrac{2\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{\frac{2}{\alpha}}\right)\left(|f'(a)|+|f'(b)|\right)\right.$$
$$+\left(\tfrac{3-\alpha}{\alpha+2}+\tfrac{\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{\frac{2-\alpha}{\alpha}}\right)\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|+\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)$$
$$\left.+\tfrac{2}{(\alpha+1)(\alpha+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|\right).$$

Corollary 2.2. *In Theorem 2.1, if we take $\alpha = 1$, then we get*

$$\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq \tfrac{b-a}{36}\left(\left(\tfrac{2s}{(s+1)(s+2)}+\tfrac{1}{(s+1)(s+2)}\left(\tfrac{1}{2}\right)^{s-1}\right)\left(|f'(a)|+|f'(b)|\right)\right.$$
$$+\left(\tfrac{3s+1}{(s+1)(s+2)}+\tfrac{1}{(s+1)(s+2)}\left(\tfrac{1}{2}\right)^{s-1}\right)\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|+\left|f'\left(\tfrac{a+2b}{3}\right)\right|\right)$$
$$\left.+\tfrac{2}{(s+1)(s+2)}\left|f'\left(\tfrac{a+b}{2}\right)\right|\right).$$

Corollary 2.3. *In Theorem 2.1, if we take $\alpha = s = 1$, then we get*

$$\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{1}{b-a}\int_a^b f(u)\,du\right|$$

$$\leq \tfrac{b-a}{12}\left(\tfrac{3|f'(a)|+5\left|f'\left(\frac{2a+b}{3}\right)\right|+2\left|f'\left(\frac{a+b}{2}\right)\right|+5\left|f'\left(\frac{a+2b}{3}\right)\right|+3|f'(b)|}{18}\right).$$

Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L^1[a, b]$ with $0 \leq a < b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s \in (0, 1]$ where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right|$$

$$\leq \tfrac{b-a}{36}\left(\left(\tfrac{2^{p-\frac{1}{\alpha}}}{\alpha}\left(B\left(\tfrac{1}{\alpha},p+1\right)+\tfrac{1}{p+1}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2,\tfrac{1}{2}\right)\right)\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\Bigg),$$

where $\mathcal{S}_\alpha$ is defined by (2.1), $B(.,.)$ and ${}_2F_1(.,.,.,.)$ are beta and hypergeometric functions respectively.

Proof. From Lemma 2.1, properties of modulus, Hölder's inequality, and s-convexity in the second sens of $|f'|^q$, we have

$$\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^\alpha}\mathcal{S}_\alpha\right|$$

$$\leq \frac{b-a}{36}\left(\left(\int_0^1|4t^\alpha-2|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)a+t\frac{2a+b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1(1-t)^{p\alpha}\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\frac{2a+b}{3}+t\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 t^{p\alpha}dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\frac{a+b}{2}+t\frac{a+2b}{3}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_0^1|4(1-t)^\alpha-2|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-t)\frac{a+2b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\right)$$

$$\leq \frac{b-a}{36}\left(\left(\int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(2-4t^\alpha)^p\,dt+\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1(4t^\alpha-2)^p\,dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1\left((1-t)^s|f'(a)|^q+t^s\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q+t^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\int_0^1\left((1-t)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+t^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\right)dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4(1-t)^\alpha-2\right)^p dt+\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1\left(2-4(1-t)^\alpha\right)^p dt\right)^{\frac{1}{p}}$$

$$\left.\times\left(\int_0^1\left((1-t)^s\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q+t^s\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right)$$

$$=\frac{b-a}{36}\left(\left(\frac{2^{p-\frac{1}{\alpha}}}{\alpha}\left(B\left(\tfrac{1}{\alpha},p+1\right)+\tfrac{1}{p+1}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2,\tfrac{1}{2}\right)\right)\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

$$\left.+\left(\tfrac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),$$

We have use the fact that

$$\int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(2-4t^\alpha\right)^p dt=\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1\left(2-4(1-t)^\alpha\right)^p dt=\tfrac{1}{\alpha}2^{p-\frac{1}{\alpha}}B\left(\tfrac{1}{\alpha},p+1\right)$$

and

$$\begin{aligned}\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1\left(4t^\alpha-2\right)^p dt &= \int_0^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}\left(4(1-t)^\alpha-2\right)^p dt\\ &= \tfrac{1}{(p+1)\alpha}2^{p-\frac{1}{\alpha}}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2,\tfrac{1}{2}\right).\end{aligned}$$

The proof is completed. □

Corollary 2.4. *In Theorem 2.2, if we take $s=1$, then we get*

$$\begin{aligned}
&\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right| \\
\leq\ & \frac{b-a}{36}\left(\left(\frac{2^{p-\frac{1}{\alpha}}}{\alpha}\left(B\left(\frac{1}{\alpha},p+1\right)+\frac{1}{p+1}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1,p+2,\frac{1}{2}\right)\right)\right)^{\frac{1}{p}}\right. \\
&\times\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right) \\
&\left.+\left(\frac{1}{p\alpha+1}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Corollary 2.5. *In Theorem 2.2, if we take $\alpha=1$, then we get*

$$\begin{aligned}
&\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\ & \frac{b-a}{36}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(2\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)\right. \\
&\left.+\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Corollary 2.6. *In Theorem 2.2, if we take $\alpha=s=1$, then we get*

$$\begin{aligned}
&\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right| \\
\leq\ & \frac{b-a}{36}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(2\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right. \\
&\left.+\left(\left(\frac{\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Theorem 2.3. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\leq a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q\geq 1$, then we have*

$$\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right|$$

$$
\begin{aligned}
\leq\ & \tfrac{b-a}{36}\left(\left(\tfrac{2-2\alpha}{\alpha+1}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\right. \\
& \times\left(\left(\left(\tfrac{2}{s+1}-\tfrac{4}{s+1}\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\mathcal{L}\left(s+1,\alpha+1\right)\right)\left|f'\left(a\right)\right|^{q}\right.\right. \\
& \left.+\left(\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-\alpha}{\alpha}}+\tfrac{2(s+1)-2\alpha}{(s+1)(\alpha+s+1)}\right)\left|f'\left(\tfrac{2a+b}{3}\right)\right|^{q}\right)^{\frac{1}{q}} \\
& +\left(\left(\left(\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-\alpha}{\alpha}}+\tfrac{2(s+1)-2\alpha}{(s+1)(\alpha+s+1)}\right)\left|f'\left(\tfrac{a+2b}{3}\right)\right|^{q}\right.\right. \\
& \left.+\left(\tfrac{2}{s+1}-\tfrac{4}{s+1}\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\mathcal{L}\left(s+1,\alpha+1\right)\right)\left|f'\left(b\right)\right|^{q}\right)^{\frac{1}{q}} \\
& +\left(\tfrac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(\tfrac{1}{\alpha+s+1}\left|f'\left(\tfrac{2a+b}{3}\right)\right|^{q}+B\left(s+1,\alpha+1\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\right. \\
& \left.\left.+\left(B\left(s+1,\alpha+1\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\tfrac{1}{\alpha+s+1}\left|f'\left(\tfrac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\right)\right),
\end{aligned}
$$

where $\mathcal{S}_\alpha$ *is defined by* (2.1).

Proof. From Lemma 2.1, properties of modulus, power mean inequality, and s-convexity in the second sens of $|f'|^q$, we have

$$
\begin{aligned}
& \left|\tfrac{1}{6}\left(f\left(a\right)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f\left(b\right)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_\alpha\right| \\
\leq\ & \tfrac{b-a}{36}\left(\left(\int_0^1\left|4t^{\alpha}-2\right|dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left|4t^{\alpha}-2\right|\left|f'\left(\left(1-t\right)a+t\tfrac{2a+b}{3}\right)\right|^{q}dt\right)^{\frac{1}{q}}\right. \\
& +\left(\int_0^1\left(1-t\right)^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1\left(1-t\right)^{\alpha}\left|f'\left(\left(1-t\right)\tfrac{2a+b}{3}+t\tfrac{a+b}{2}\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1 t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t^{\alpha}\left|f'\left(\left(1-t\right)\tfrac{a+b}{2}+t\tfrac{a+2b}{3}\right)\right|^{q}dt\right)^{\frac{1}{q}} \\
& +\left(\int_0^1\left|4\left(1-t\right)^{\alpha}-2\right|dt\right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left(\int_0^1 |4(1-t)^\alpha-2|\left|f'\left((1-t)\tfrac{a+2b}{3}+tb\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \tfrac{b-a}{36}\Bigg(\left(\tfrac{2-2\alpha}{\alpha+1}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg(|f'(a)|^q\int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(2-4t^\alpha)(1-t)^s\,dt+\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q\int_0^{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(2-4t^\alpha)t^s dt$$

$$+|f'(a)|^q\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1(4t^\alpha-2)(1-t)^s\,ds+\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q\int_{\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1(4t^\alpha-2)t^s dt\Bigg)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f'\left(\tfrac{2a+b}{3}\right)\right|^q\int_0^1(1-t)^{\alpha+s}\,dt+\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1(1-t)^\alpha t^s dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\int_0^1 t^\alpha(1-t)^s\,dt+\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\int_0^1 t^{\alpha+s}dt\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{2-2\alpha}{\alpha+1}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg(\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\int_0^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(4(1-t)^\alpha-2)(1-t)^s\,dt$$

$$+|f'(b)|^q\int_0^{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}(4(1-t)^\alpha-2)t^s dt$$

$$+\left|f'\left(\tfrac{a+2b}{3}\right)\right|^q\int_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^1(2-4(1-t)^\alpha)(1-t)^s\,dt$$

$$+\left|f'(b)\right|^{q}\int\limits_{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha}}}^{1}\left(2-4(1-t)^{\alpha}\right)t^{s}dt\Bigg)^{\frac{1}{q}}\Bigg)$$

$$\begin{aligned}
&= \tfrac{b-a}{36}\Bigg(\left(\tfrac{2-2\alpha}{\alpha+1}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\Bigg(\Bigg(\tfrac{2}{s+1}-\tfrac{4}{s+1}\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\mathcal{L}(s+1,\alpha+1)\Bigg)\left|f'(a)\right|^{q} \\
&+\left(\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-\alpha}{\alpha}}+\tfrac{2(s+1)-2\alpha}{(s+1)(\alpha+s+1)}\right)\left|f'\left(\tfrac{2a+b}{3}\right)\right|^{q}\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\Bigg(\left(\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{2}\right)^{\frac{s+1-\alpha}{\alpha}}+\tfrac{2(s+1)-2\alpha}{(s+1)(\alpha+s+1)}\right)\left|f'\left(\tfrac{a+2b}{3}\right)\right|^{q} \\
&+\left(\tfrac{2}{s+1}-\tfrac{4}{s+1}\left(1-\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\mathcal{L}(s+1,\alpha+1)\right)\left|f'(b)\right|^{q}\Bigg)^{\frac{1}{q}} \\
&+\left(\tfrac{1}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg(\left(\tfrac{1}{\alpha+s+1}\left|f'\left(\tfrac{2a+b}{3}\right)\right|^{q}+B(s+1,\alpha+1)\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}} \\
&+\left(B(s+1,\alpha+1)\left|f'\left(\tfrac{a+b}{2}\right)\right|^{q}+\tfrac{1}{\alpha+s+1}\left|f'\left(\tfrac{a+2b}{3}\right)\right|^{q}\right)^{\frac{1}{q}}\Bigg)\Bigg),
\end{aligned}$$

where we have used (2.8)-(2.13). The proof is achieved. □

Corollary 2.7. *In Theorem 2.3, if we take $s=1$, then we get*

$$\begin{aligned}
&\left|\tfrac{1}{6}\left(f(a)+2f\left(\tfrac{2a+b}{3}\right)+2f\left(\tfrac{a+2b}{3}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{3^{1-\alpha}(b-a)^{\alpha}}\mathcal{S}_{\alpha}\right| \\
\le\; & \tfrac{b-a}{36}\Bigg(\left(\tfrac{2-2\alpha}{\alpha+1}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}\right)^{1-\frac{1}{q}} \\
&\times\Bigg(\Bigg(\left(\tfrac{4-(\alpha+1)(\alpha+2)}{(\alpha+1)(\alpha+2)}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}-\tfrac{2\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{\frac{2}{\alpha}}\right)\left|f'(a)\right|^{q} \\
&+\left(\tfrac{\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{\frac{2-\alpha}{\alpha}}+\tfrac{2-\alpha}{\alpha+2}\right)\left|f'\left(\tfrac{2a+b}{3}\right)\right|^{q}\Bigg)^{\frac{1}{q}} \\
&+\Bigg(\Bigg(\left(\tfrac{\alpha}{(\alpha+2)}\left(\tfrac{1}{2}\right)^{\frac{2-\alpha}{\alpha}}+\tfrac{2-\alpha}{\alpha+2}\right)\left|f'\left(\tfrac{a+2b}{3}\right)\right|^{q} \\
&+\left(\tfrac{4-(\alpha+1)(\alpha+2)}{(\alpha+1)(\alpha+2)}+\tfrac{4\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{\frac{1}{\alpha}}-\tfrac{2\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{\frac{2}{\alpha}}\right)\left|f'(b)\right|^{q}\Bigg)^{\frac{1}{q}} \\
&+\left(\tfrac{1}{\alpha+1}\right)\Bigg(\left(\tfrac{(\alpha+1)\left|f'\left(\frac{2a+b}{3}\right)\right|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{\alpha+2}\right)^{\frac{1}{q}}+\left(\tfrac{(\alpha+1)\left|f'\left(\frac{a+2b}{3}\right)\right|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{\alpha+2}\right)^{\frac{1}{q}}\Bigg)\Bigg).
\end{aligned}$$

Corollary 2.8. *In Theorem 2.3, if we take $\alpha = 1$, then we get*

$$\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{36}\left(\left(\frac{2s}{(s+1)(s+2)}+\frac{1}{(s+1)(s+2)}\left(\frac{1}{2}\right)^{s-1}\right)^{\frac{1}{q}}\right.$$
$$\times\left(\left(|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right)^{\frac{1}{q}}+\left(\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q\right)^{\frac{1}{q}}\right)$$
$$\left.+\frac{1}{2}\left(\left(\frac{2(s+1)\left|f'\left(\frac{2a+b}{3}\right)\right|^q+2\left|f'\left(\frac{a+b}{2}\right)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+b}{2}\right)\right|^q+2(s+1)\left|f'\left(\frac{a+2b}{3}\right)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.9. *In Theorem 2.3, if we take $\alpha = s = 1$, then we get*

$$\left|\frac{1}{6}\left(f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{36}\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{2a+b}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+2b}{3}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{2}\left(\left(\frac{2\left|f'\left(\frac{2a+b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+2b}{3}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{3}\right)^{\frac{1}{q}}\right)\right).$$

3. Applications to special means

For arbitrary real numbers $a, a_1, a_2, ..., a_n, b$ we have:

The Arithmetic mean: $A(a_1, a_2, ..., a_n) = \frac{a_1+a_2+...+a_n}{2}$.

The p-Logarithmic mean: $L_p(a,b) = \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}$, $a, b > 0, a \neq b$ and $p \in \mathbb{R}\setminus\{-1, 0\}$.

Proposition 3.1. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\left|A\left(a^2, b^2\right) + A^2(a, a, b) + A^2(a, b, b) - 3L_2^2(a, b)\right| \leq \frac{b^2-a^2}{4}.$$

Proof. The assertion follows from Corollary 2.3, applied to the function $f(x) = x^2$.

□

Proposition 3.2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$\begin{aligned} \frac{4}{15} \quad & \left| A\left(a^{\frac{5}{4}}, b^{\frac{5}{4}}\right) + A^{\frac{5}{4}}(a,a,b) + A^{\frac{5}{4}}(a,b,b) - 3L_{\frac{4}{5}}^{\frac{4}{5}}(a,b) \right| \\ \leq \; & \tfrac{5\sqrt{2}(b-a)}{144} \left(2\left(\left(a^{\frac{1}{2}} + \left(\tfrac{2a+b}{3}\right)^{\frac{1}{2}} \right)^{\frac{1}{2}} + \left(\left(\tfrac{a+2b}{3}\right)^{\frac{1}{2}} + b^{\frac{1}{2}} \right)^{\frac{1}{2}} \right) \right. \\ & \left. + \left(\left(\tfrac{2a+b}{3}\right)^{\frac{1}{2}} + \left(\tfrac{a+b}{2}\right)^{\frac{1}{2}} \right)^{\frac{1}{2}} + \left(\left(\tfrac{a+b}{2}\right)^{\frac{1}{2}} + \left(\tfrac{a+2b}{3}\right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \right). \end{aligned}$$

Proof. The assertion follows from Corollary 2.5 with $q = 2$, applied to the function $f(x) = \frac{4}{5}x^{\frac{5}{4}}$, which the derivatives $|f'(x)|^2 = x^{\frac{1}{2}}$ is $\frac{1}{2}$-convex. □

4. Conclusion

In this study, we have considered the Simpson like type integral inequalities, which the main results of the paper can be summarized as follows:

(1) A new identity regarding Simpson like type inequalities is proved.
(2) Some new fractional Simpson like type inequalities for functions whose first derivatives are s-convex are established.
(3) Some special cases are derived.
(4) Applications of our findings are provided.

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