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MULTIPLICATION Γ-SEMIGROUPS

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Abstract. In this paper, we define the concept of non-commutative right multiplication Γ-semigroup as a generalization of non-commutative right multiplication semigroups using right Γ-ideals. We prove some results related to regular Γ-semigroups, simple Γ-semigroups, semisimple Γ-semigroups and cancellative Γ-semigroups.

1. Introduction

The concept of multiplication ring was introduced for developing an algebraic structure endowed with the notion of factorisation of ideals. It has been extensively studied by several authors and powerful ideal-theoretic techniques have been employed to characterise them [3]. With the rapid development of semigroup theory (see [4, 5]), a parallel line of research work was initiated in the field of semigroup and the idea of multiplication semigroup was introduced by Mannepalli [1]. A commutative semigroup S is called a multiplication semigroup if for any two ideals A, B of S such that $A \subset B$, there exists an ideal C of S such that $A = BC$. In [1], the structure of multiplication semigroup was depicted with identity in commutative case. For more details on multiplication semigroup, the reader is referred to [2].

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In 1986, Sen and Saha [8] introduced the notion of Γ-semigroups as a generalization of semigroups. The motivation for this concept is such that in any semigroup X, the operation on X is certainly associative on every subset T of X but not necessarily closed. Many classical notions such as ideals in semigroups, regular semigroups, simple semigroups, semisimple and cancellative semigroups have been generalized to Γ-semigroups (see [7, 8]).

Generalization plays a very useful role in mathematics. In non-commutative right multiplication semigroups, non-commutative right multiplication Γ-semigroups are one of their generalizations in that the class of all non-commutative right multiplication Γ-semigroups includes non-commutative right multiplication semigroups. Thus, inspired by the research work of [1], we introduce the concept of right multiplication Γ-semigroup in non-commutative case and prove some properties enjoyed by some classes of Γ-semigroups.

2. Preliminaries

In this section, we recall some definitions needed for this paper from [1], [8] and [9].

Definition 2.1. Let $S = \{a, b, c...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two non-empty sets. Then S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S \mid (a, \alpha, b) \longrightarrow a\alpha b \in S$ satisfying $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Let S be an arbitrary semigroup and Γ any non-empty set. Define $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. Then S is a Γ-semigroup. Thus, a semigroup can be considered to be a Γ-semigroup.

Let S be a Γ-semigroup. A non-empty subset A of Γ-semigroup S is said to be a Γ-subsemigroup of S if $A\Gamma A \subseteq A$. If A and B are two non-empty subsets of a Γ-semigroup S, then $A\Gamma B$ is defined as $A\Gamma B = \{a\gamma b \mid a \in A,\ b \in B \text{ and } \gamma \in \Gamma\}$.

For simplicity we write $a\Gamma B$, $A\Gamma b$ and $A\gamma B$ in place of $\{a\}\Gamma B$, $A\Gamma\{b\}$ and $A\{\gamma\}B$ respectively.

Definition 2.2. A non-empty subset A of a Γ-semigroup S is called a left (right) Γ-ideal (or simply a left (right) ideal of S if $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$). Further, a non-empty subset A of a Γ-semigroup S is called a Γ-ideal if A is both a left and a right Γ-ideal of S. By a proper left (right) Γ-ideal of S, we mean a left (right) Γ-ideal A of S such that $A \neq S$.

Definition 2.3. A proper left (right) Γ-ideal A of S is called a maximal left (right) Γ-ideal of S if A is not contained in any proper left (right) Γ-ideal of S.

Definition 2.4. A left (right) Γ-ideal A of a Γ-semigroup S is called idempotent if $A\Gamma A = A$.

Definition 2.5. Let S be a Γ-semigroup. For each element a of a Γ-semigroup S, the left Γ-ideal $a \cup S\Gamma a$ containing a is the smallest left Γ-ideal of S containing a. If A is any other left Γ-ideal containing a, then $a \cup S\Gamma a \subseteq A$. This is denoted by $\langle a\rangle_l$ and called the principal left Γ-ideal generated by the element a. Similarly, for each $a \in S$, the smallest right Γ-ideal containing a is $a \cup a\Gamma S$ which is denoted by $\langle a\rangle_r$ and called the principal right Γ-ideal generated by the element a. The principal Γ-ideal of S generated by the element a is denoted by $\langle a\rangle$ and $\langle a\rangle = a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$.

Definition 2.6. A Γ-semigroup S is said to be left (right) cancellative if $a\alpha x = a\alpha y \Rightarrow x = y$ ($x\beta b = y\beta b \Rightarrow x = y$) for all $a, b, x, y \in S$, $\alpha, \beta \in \Gamma$. A Γ-semigroup S is called cancellative if S is both left and right Γ-cancellative.

Definition 2.7. A Γ-semigroup S is said to be regular if for each $a \in S$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.8. A Γ-semigroup S is said to be (left, right, intra-) regular if for each $a \in S$, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that ($a = x\alpha a\beta a,\ a = a\alpha a\beta x,\ a = x\alpha a\beta a\gamma y$).

Definition 2.9. Let S be a Γ-semigroup. An element a of a Γ-semigroup S is said to be an α-idempotent if $a\alpha a = a$ for some $\alpha \in \Gamma$. The set of all α-idempotents is denoted by E_α. We denote $\bigcup_{\alpha\in\Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of S. If $S = E(S)$, then S is called an idempotent Γ-semigroup.

Definition 2.10. A Γ-semigroup S is called left (right) simple if it contains no proper left (right) Γ-ideal, i.e., for every $a \in S,\ S\Gamma a = S\ (a\Gamma S = S)$.

Definition 2.11. A Γ-semigroup S is said to be semisimple if $\langle a\rangle \Gamma \langle a\rangle = \langle a\rangle$ for each element a of S.

Definition 2.12. Let S be a non-commmutative semigroup. Then S is called a right (left) multiplication semigroup if for any pair of right (left) ideals A and B with $A \subseteq B$ there exists a right (left) ideal C such that $A = CB$ ($A = BC$).

3. Multiplication Γ-semigroups

Definition 3.1. Let S be a non-commutative Γ-semigroup. Then S is called a right multiplication Γ-semigroup if for any two right Γ-ideals A, B of S such that $A \subseteq B$, there exists a right Γ-ideal C of S such that $A = C\Gamma B$. Dually, left multiplication Γ-semigroups can be defined.

Example 3.1. *Let $S = \{1, -1, i, -i\}$ be a semigroup under the operation given by the table below:*

Let $\Gamma = \{\alpha\}$. Define $a\alpha b = ab$. As $a\alpha(b\alpha c) = (a\alpha b)\alpha c$ for all $a, b, c \in S$, S is a Γ-semigroup. Let $A = \{1\}$, $B = \{1, -i\}$, $C = \{1, i\}$. Then A, B and C are all right Γ-ideals of the Γ-semigroup S. It is not difficult to verify that S is a right

.	1	−1	i	$-i$
1	1	1	1	1
−1	1	−1	−1	1
i	1	i	i	1
$-i$	$-i$	$-i$	$-i$	$-i$

multiplication Γ*-semigroup. However, suppose* $A = \{1\}$*,* $B = \{1, -i\}$*,* $C = \{-i\}$*. Evidently,* A*,* B *and* C *are all right* Γ*-ideals of* S*. It may be easily seen that* S *is not a right multiplication* Γ*-semigroup.*

Remark 1. *In a right multiplication* Γ*-semigroup* S*,* $A = A\Gamma B$ *for every right* Γ*-ideal* A *of* S*.*

Proposition 3.1. *Let* S *be a right multiplication* Γ*-semigroup. Then*

(i) $S\Gamma S = S$.

(ii) $x \in x\Gamma S$ *for each* $x \in S$*.*

Proof. (i) Since $S \subseteq S$, we have $S = A\Gamma S$ for some right Γ-ideal A and thus $S\Gamma S = S$. (ii) For any $x \in S$, $\{x\} \cup x\Gamma S \subseteq S$ so that for some right Γ-ideal B of S we have $\{x\} \cup x\Gamma S = B\Gamma S = B\Gamma S\Gamma S = (\{x\} \cup x\Gamma S)\Gamma S = x\Gamma S$. Hence, $x \in x\Gamma S$ for each $x \in S$. □

The following result is an analogue of [[2], Propostion 1.16(ii)].

Proposition 3.2. *Let* S *be right multiplication* Γ*-semigroup. Then* $A \subseteq S\Gamma A$ *for any right* Γ*-ideal* A *of* S*.*

Proof. Let A be any right Γ-ideal of S. Since $A \subseteq A$, we have $A = B\Gamma A$ for some right Γ-ideal B of S and thus $A \subseteq S\Gamma A$. □

The following result is an analogue of [[2], Propostion 1.16(iii)].

Proposition 3.3. *Let S be a right multiplication Γ-semigroup. If M is a maximal right Γ-ideal containing every proper right Γ-ideal of S, then $M = M\Gamma M$ or M is a Γ-ideal of S such that $M = S\Gamma M$.*

Proof. Let M be a maximal right Γ-ideal of S. Since $M \subseteq M$, we have $M = A\Gamma M$ for some right Γ-ideal A of S. If $A \subseteq M$, then $M = M\Gamma M$. Suppose $A \nsubseteq M$. Then $A \cup M = S$. Now, $A\Gamma M \cup M\Gamma M = S\Gamma M$ i.e., $M = S\Gamma M$. Thus M is a left Γ-ideal of S. Also, $M\Gamma M \cup M\Gamma A = M\Gamma S$ i.e., $M = M\Gamma S$. Hence, M is a right Γ-ideal of S. □

The following result is an analogue of [[2], Propostion 1.16(iii)].

Proposition 3.4. *Let S be a right multiplication Γ-semigroup and M be a maximal right Γ-ideal containing every proper right Γ-ideal of S. If M is unique such that $M \neq M\Gamma M$, then $M = x\Gamma S$ for some $x \in M \setminus M\Gamma M$.*

Proof. Let M be unique such that $M \neq M\Gamma M$. For any $x \in M \setminus M\Gamma M$ and since $x\Gamma M \subseteq M$ by Proposition 2.3, we have $x\Gamma M = A\Gamma M$ for some right Γ-ideal A of S. Clearly, $A = S$ since otherwise $x \in M\Gamma M$ which is not true. Thus, $x\Gamma S = S\Gamma M = M$. □

The following result is an analogue of [[2], Propostion 1.16(viii)].

Proposition 3.5. *Let S be a right multiplication Γ-semigroup. If S contains a left cancellative element, then S contains a β-idempotent which is a left identity.*

Proof. Let $a \in S$ be a left cancellative element. By Proposition 3.1 (ii), $a = a\alpha s$ for some $s \in S$ and $\alpha \in \Gamma$, so that $a\alpha s = a\alpha s\beta s$ and thus $s = s\beta s$ since a is left cancellative. Suppose $s\beta x = s\beta y$ for some $x, y \in S$ and $\beta \in \Gamma$. Then $a\alpha s\beta x = a\alpha s\beta y$, i.e., $a\beta x = a\beta y$ and hence $x = y$. This shows that s is a left cancellative element. Now, for any $b \in S$ and $\beta, \gamma \in \Gamma$ we have $s\beta b = s\beta s\gamma b$. Then $b = s\gamma b$. Thus, the β-idempotent element s is left identity in S. □

Proposition 3.6. *Let S be a right multiplication Γ-semigroup. If S is left Γ- and right Γ-cancellative, then S contains an identity and every right Γ-ideal is a Γ-ideal.*

Proof. Since S is left cancellative, S contains a left identity by Proposition 3.5. If $s \in S$ is a left identity element in S, then $a\alpha s = a\alpha s\beta s$ for every $a \in S$ and $\alpha, \beta \in \Gamma$. Since S is right cancellative, $a = a\alpha s$ shows that s is also a right identity in S. Now, let A be any Γ-ideal in S. For any $x \in A$, $\{x\} \cup x\Gamma S = C\Gamma\{x\} \cup x\Gamma S$ for some right Γ-ideal C of S. It follows that $x = a\alpha x$ or $a\alpha x\beta b$ for some $a \in C,\ b \in S$ and $\alpha, \beta \in \Gamma$. If $x = a\alpha x$, then $s\alpha x = x = a\alpha x$ and so $s = a$ since S is right cancellative. Now, $s = a \in C$ and thus $C = S$. We now have $\{x\}\Gamma\{x\} \cup x\Gamma S = S\Gamma(x \cup x\Gamma S)$ which shows that $S\Gamma A \subseteq A$ i.e., A is a Γ-ideal of S in this case. Suppose $x = a\alpha x\beta b$. Since $a \in C$, $a\Gamma S \subseteq C$ and so $a\Gamma S\Gamma x \subseteq \{x\} \cup x\Gamma S$. For any $u \in S$ we have $a\alpha u\beta x = x$ or $x\gamma u^*$, $u^* \in S$ and $\gamma \in \Gamma$. In the first case, $a\alpha u\beta x = x = a\alpha x\beta b$ and so $u\beta x = x\beta b \in A$ since S is left Γ-cancellative. In the second case, $a\alpha u\beta x = x\gamma u^* = a\alpha x\beta b\gamma u^*$ and so $u\beta x = x\beta b\gamma u^* \in A$. In either case $S\Gamma x \subseteq A$, which shows that $S\Gamma A \subseteq A$. Thus, A is a Γ-ideal of S. □

4. Semisimplicity in multiplication Γ-semigroups

In this section, we show that simple Γ-semigroups, regular Γ-semigroups and right regular Γ-semigroups are right multiplication Γ-semigroups. Also, we obtain a condition for semisimple Γ-semigroups to be right multiplication Γ-semigroups.

The following result is an analogue of [[2], Propostion 1.19].

Proposition 4.1. *Let S be a Γ-semigroup. If $a \in (a\Gamma S)\Gamma(a\Gamma S)$ for every $a \in S$, then S is a right multiplication Γ-semigroup.*

Proof. Let A and B be any two right Γ-ideals of S such that $A \subseteq B$. For every $a \in A$, we have $a \in (a\Gamma S)\Gamma(a\Gamma S) \subseteq A\Gamma B$. Thus, $A \subseteq A\Gamma B \subseteq A$, i.e., $A = A\Gamma B$ which shows that S is a right multiplication Γ-semigroup. □

The following result is an analogue of [[2], Corollary 1.20].

Proposition 4.2. *Let S be a Γ-semigroup. If S is regular, then S is a right multiplication Γ-semigroup.*

Proof. Suppose S is a regular Γ-semigroup. Then for every $a \in S$ there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. Clearly, $a \in (a\Gamma S)\Gamma(a\Gamma S)$. Now the result follows from Proposition 4.1. □

Proposition 4.3. *Let S be a Γ-semigroup. If S is right regular, then S is a right multiplication Γ-semigroup.*

Proof. Similar to the proof of Proposition 4.2. □

Proposition 4.4. *Let S be a Γ-semigroup. If S is simple with $a \in a\Gamma S$ for every $a \in S$, then S is a right multiplication Γ-semigroup.*

Proof. Let S be a simple semigroup with $a \in a\Gamma S$ for every $a \in S$. Let A and B be any two right Γ-ideals of S such that $A \subseteq B$. Now, $A\Gamma S \subseteq A$ and consequently $A\Gamma S\Gamma B \subseteq A\Gamma B$. Since S is simple, we have $S\Gamma B = S$. Thus, $A \subseteq A\Gamma S \subseteq A\Gamma B \subseteq A$. Hence, $A = A\Gamma B$. This shows that S is a right multiplication Γ-semigroup. □

The following result is an analogue of [[2], Theorem 1.21].

Proposition 4.5. *Let S be a Γ-semigroup. If S is a left simple right multiplication Γ-semigroup, then S is a left simple Γ-semigroup containing idempotents.*

Proof. Let S be a right multiplication Γ-semigroup. For any $a \in S$, we have $a = a\alpha x$ for some $x \in S$ and $\alpha \in \Gamma$ by Proposition 3.1 (ii). Since S is left simple, $a \in S = S\Gamma a$. Therefore, $a \in a\Gamma S\Gamma a$. Hence, S contains idempotents. □

Proposition 4.6. *Let S be a right multiplication Γ-semigroup. If S is intra-regular, then S is semisimple.*

Proof. Let $\langle a\rangle = a \cup S\Gamma a \cup a\Gamma S \cup S\Gamma a\Gamma S$ be the smallest Γ-ideal of S containing a. Since S is intra-regular and for any $a \in S$, $a \in S\Gamma a\Gamma a\Gamma S = (S\Gamma a)\Gamma(a\Gamma S) \subseteq \langle a\rangle \Gamma \langle a\rangle$ i.e., $\langle a\rangle = \langle a\rangle \Gamma \langle a\rangle$. Hence, S is semisimple. □

The following result is an analogue of [[2], Theorem 1.23].

Proposition 4.7. *Suppose that S is a Γ-semigroup such that every right Γ-ideal is a Γ-ideal. Then S is a semisimple right multiplication Γ-semigroup if and only if S is right regular.*

Proof. Let S be a semisimple right multiplication Γ-semigroup. Then for every $a \in S$ we have $a \in (a\Gamma S)\Gamma(a\Gamma S)$. Since every right Γ-ideal is Γ-ideal, it follows that $a \in a\Gamma S\Gamma a\Gamma S \subseteq a\Gamma a\Gamma S$.
Conversely, suppose S is right regular. Then for any $a \in S$, $a \in a\Gamma a\Gamma S = a\Gamma(a\Gamma S) \subseteq \langle a\rangle \Gamma \langle a\rangle$ i.e., $\langle a\rangle = \langle a\rangle \Gamma \langle a\rangle$. Thus, S is semisimple. □

Proposition 4.8. *Let S be a left cancellative Γ-semigroup. If S is semisimple right multiplication Γ-semigroup, then S is a simple Γ-semigroup such that $a \in a\Gamma S$ for each $a \in S$.*

Proof. Suppose $a \in \langle a\rangle \Gamma \langle a\rangle$. Then $a = a\alpha x\beta a\gamma y \in a\Gamma S\Gamma a\Gamma S$ for some $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Since S is left cancellative, it follows that $x\beta a\gamma y$ is α-idempotent which is also a left identity of S. But $x\beta a\gamma y \in \langle a\rangle \Gamma \langle a\rangle$ shows that $\langle a\rangle = S$ which is not true. Hence, S is simple. By Proposition 3.1 (ii), we have $a \in a\Gamma S$ for each $a \in S$. □

It is necessary to note that the results presented in this paper for right multiplication Γ-semigroups can also be carried out in a similar manner for left multiplication Γ-semigroups.

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