Jordan Journal of Mathematics and Statistics (JJMS), 16(3), 2023, pp 483 - 506
DOI: https://doi.org/10.47013/16.3.7

FRACTIONAL MACLAURIN TYPE INEQUALITIES FOR FUNCTIONS WHOSE FIRST DERIVATIVES ARE s-CONVEX FUNCTIONS

S. DJENAOUI[(1)] AND B. MEFTAH[(2)]

Abstract. Classical and fractional integral inequalities have become a popular method and a powerful tool for estimating errors of quadrature formulas. Several studies on various types of inequality have been conducted and the literature in this area is vast and diverse. The current study intends to investigate one of the open three-point Newton-Cotes formulae, known as Maclaurin's formula, using Riemann-Liouville fractional operators. To accomplish so, we first created a new identity. From this identity and through the s-convexity, we have established some new Maclaurin-type inequalities, we also discussed the cases that can be derived of our finding. Furthermore, various applications for error estimates are offered to demonstrate the efficacy of our primary results.

1. Introduction

Convexity is an important and central topic in several fields, including economics, finance, optimization, and game theory. Let us review the definition of this useful concept.

Definition 1.1. [25] A function $f : I \to \mathbb{R}$ is said to be convex, if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1.1}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

If the function f is concave, then (1.1) holds in the reverse direction.

2010 *Mathematics Subject Classification.* 26D10, 26D15, 26A51.

Key words and phrases. Maclaurin's formulae, Newton-Cotes quadrature, s-convex functions,...

Received: July 3, 2022 Accepted: Oct. 30, 2022 .

It's clear that convexity and the development of the theory of inequalities go hand in hand, and we can see right away that one depends on the other. Any function that works with (1.1) must also work with the famous Hermite-Hadamard inequality, which can be written as follows: For every convex function f on the interval $[a, b]$ with $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The concept of convexity enjoys a wide field of applications. Because it is so important, there have been many generalisations, extensions, and new classes of convex functions. One of the most important is s-convexity, which Breckner came up with and defines as follows:

Definition 1.2. [6] A nonnegative function $f : I \subset [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Regarding some appeared papers dealing convexity and integral inequalities we refer readers to [4, 8, 9, 13, 16, 18, 20, 21, 22, 24] and references therein.

Nowadays, fractional calculus has become a popular tool for scientists. It has been successfully used in various fields of science and engineering see [11, 15]. Its main strength in the description of memory and genetic properties of different materials and processes has aroused great interest for researchers in different fields.

In the problems that have been written about, fractional operators like Caputo, Hadamard, Katugampola, etc., can be used. But the Riemann-Liouville operator is employed the most. Here's how to define it:

Definition 1.3. [15] Let $f \in L^1[a,b]$. The Riemann-Liouville fractional integrals $I_{a^+}^{\alpha} f$ and $I_{b^-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

respectively, where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t} t^{\alpha-1} dt$, is the gamma function and $I_{a^+}^0 f(x) = I_{b^-}^0 f(x) = f(x)$.

In [29], Sarikaya et al., used the above mentioned operator and established the fractional analogue of inequality (1.2).

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be an integrable and positive function with $0 \leq a < b$. If f is a convex function on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{a^+}^{\alpha} f(b) + I_{b^-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}.$$

Chen and Huan [7], gave some Simpson inequalities for differentiable s-convex functions via Riemann-Liouville fractional integral.

Theorem 1.2. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L^1[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f'|$ is s-convex on $[a,b]$, for some fixed $s \in (0,1]$, then the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right) - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I_{a^+}^{\alpha} f\left(\frac{a+b}{2}\right) + I_{b^-}^{\alpha} f\left(\frac{a+b}{2}\right)\right]\right| \\ \leq\ & \frac{b-a}{2^{s+1}}\left(|f'(a)| + |f'(b)|\right)(I_3 + I_4), \end{aligned}$$

where $I_3 = \int\limits_0^{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}} \left(\frac{1}{3} - \frac{t^{\alpha}}{2}\right)\left[(1+t)^s + (1-t)^s\right] dt$

and $I_4 = \int\limits_{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}}^{1} \left(\frac{t^{\alpha}}{2} - \frac{1}{3}\right)\left[(1+t)^s + (1-t)^s\right] dt.$

Theorem 1.3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L^1[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f'|^q$ is s-convex on $[a,b]$, for some fixed*

$s\in(0,1]$ and $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\begin{aligned}&\left|\tfrac{1}{5}\left(f(a)+4f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[I_{a^+}^{\alpha}f\left(\tfrac{a+b}{2}\right)+I_{b^-}^{\alpha}f\left(\tfrac{a+b}{2}\right)\right]\right|\\ \leq\ &\tfrac{b-a}{2}\left(\int\limits_0^{\left(\frac{2}{3}\right)^{\frac{1}{\alpha}}}\left|\tfrac{t^{\alpha}}{2}-\tfrac{1}{3}\right|^p dt\right)^{\frac{1}{p}}\left(\left(\tfrac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

More recently, Kamouche et al. [12], discussed the following Simpson-like inequalities for s-convex functions differentiable via the Riemann-Liouville fractional integral.

Theorem 1.4. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If $|f'|$ is s-convex in the second sense for some fixed $s\in(0,1]$, then we have*

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)+I_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)\right)\right|\\ \leq\ &\tfrac{b-a}{4}\left(\Theta_{s,\alpha}|f'(a)|\right.\\ &\left.+\left(\tfrac{\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{1}{4}\right)^{\frac{s+1}{\alpha}}+\tfrac{3s+3-\alpha}{2(s+1)(\alpha+s+1)}\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|+\Theta_{s,\alpha}|f'(b)|\right),\end{aligned}$$

where

$$\begin{aligned}\Theta_{s,\alpha}\ =\ &\tfrac{1}{4(s+1)}\left(1-2\left(1-\left(\tfrac{1}{4}\right)^{\frac{1}{\alpha}}\right)^{s+1}\right)\\ &+B_{1-\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)-B_{\left(\frac{1}{4}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1).\end{aligned}$$

Theorem 1.5. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L^1[a,b]$ with $0\le a<b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s\in(0,1]$ where $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\begin{aligned}&\left|\tfrac{1}{8}\left(f(a)+6f\left(\tfrac{a+b}{2}\right)+f(b)\right)-\tfrac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left(I_{\left(\frac{a+b}{2}\right)^-}^{\alpha}f(a)+I_{\left(\frac{a+b}{2}\right)^+}^{\alpha}f(b)\right)\right|\\ \leq\ &\tfrac{b-a}{4}\left(\tfrac{1}{4^{p+\frac{1}{\alpha}}\alpha}B\left(\tfrac{1}{\alpha},p+1\right)+\tfrac{3^{p+1}}{4^{p+1}\alpha(p+1)}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2;\tfrac{3}{4}\right)\right)^{\frac{1}{p}}\\ &\times\left(\left(\tfrac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right).\end{aligned}$$

With regard to other articles dealing with fractional inequalities, we refer readers to [1, 2, 5, 10, 17, 19, 23, 27, 28].

Motivated by the preceding findings, we propose in this paper to investigate the Maclaurin's inequality (see [1, 26]), which can be phrased as follows:

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int\limits_a^b f(u)\,du\right|\le\frac{7(b-a)^4}{51840}\left\|f^{(4)}\right\|_\infty,$$

where f is four-times continuously differentiable function on (a,b), and $\left\|f^{(4)}\right\|_\infty=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|$.

To do so, we first establish a new fractional identity. On the basis of this identity we derive some new Maclaurin-type inequalities for functions whose first derivatives are s-convex via Riemann-Liouville fractional operators. Applications in numerical integration are also presented.

2. Main results

In order to prove our results, we need the following lemma and some definitions

Lemma 2.1. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, and $f'\in L^1[a,b]$, then the following equality holds*

$$\begin{aligned}
&\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Omega_I^\alpha\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\\
=\ &\frac{b-a}{9}\left(\int\limits_0^1\frac{1}{4}\varkappa^\alpha f'\left((1-\varkappa)a+\varkappa\frac{5a+b}{6}\right)d\varkappa\right.\\
&+\int\limits_0^1\left(\frac{3}{8}-(1-\varkappa)^\alpha\right)f'\left((1-\varkappa)\frac{5a+b}{6}+\varkappa\frac{a+b}{2}\right)d\varkappa\\
&-\int\limits_0^1\left(\frac{3}{8}-\varkappa^\alpha\right)f'\left((1-\varkappa)\frac{a+b}{2}+\varkappa\frac{a+5b}{6}\right)d\varkappa\\
&\left.-\int\limits_0^1\frac{1}{4}(1-\varkappa)^\alpha f'\left((1-\varkappa)\frac{a+5b}{6}+\varkappa b\right)d\varkappa\right),
\end{aligned}$$

where

$$\begin{aligned}
\Omega_I^\alpha\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)=&\left(I^\alpha_{\left(\frac{5a+b}{6}\right)^-}f(a)+I^\alpha_{\left(\frac{a+5b}{6}\right)^+}f(b)\right)\\
&+\frac{1}{2^{\alpha-1}}\left(I^\alpha_{\left(\frac{5a+b}{6}\right)^+}f\left(\frac{a+b}{2}\right)+I^\alpha_{\left(\frac{a+5b}{6}\right)^-}f\left(\frac{a+b}{2}\right)\right).
\end{aligned}\tag{2.1}$$

Proof. Let

$$I = I_1 + I_2 - I_3 - I_4, \tag{2.2}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \tfrac{1}{4}\varkappa^{\alpha} f'\left((1-\varkappa)\, a + \varkappa \tfrac{5a+b}{6}\right) d\varkappa, \\ I_2 &= \int_0^1 \left(\tfrac{3}{8} - (1-\varkappa)^{\alpha}\right) f'\left((1-\varkappa)\, \tfrac{5a+b}{6} + \varkappa \tfrac{a+b}{2}\right) d\varkappa, \\ I_3 &= \int_0^1 \left(\tfrac{3}{8} - \varkappa^{\alpha}\right) f'\left((1-\varkappa)\, \tfrac{a+b}{2} + \varkappa \tfrac{a+5b}{6}\right) d\varkappa \end{aligned}$$

and

$$I_4 = \int_0^1 \tfrac{1}{4}\, (1-\varkappa)^{\alpha}\, f'\left((1-\varkappa)\, \tfrac{a+5b}{6} + \varkappa b\right) d\varkappa.$$

Integrating by parts I_1, we get

$$\begin{aligned} I_1 &= \tfrac{3}{2(b-a)} \varkappa^{\alpha} f\left((1-\varkappa)\, a + \varkappa \tfrac{5a+b}{6}\right)\Big|_{\varkappa=0}^{\varkappa=1} \\ &\quad - \tfrac{3\alpha}{2(b-a)} \int_0^1 \varkappa^{\alpha-1} f\left((1-\varkappa)\, a + \varkappa \tfrac{5a+b}{6}\right) d\varkappa \\ &= \tfrac{3}{2(b-a)} f\left(\tfrac{5a+b}{6}\right) - \tfrac{6^{\alpha+1}\alpha}{4(b-a)^{\alpha+1}} \int_a^{\frac{5a+b}{6}} (\varpi - a)^{\alpha-1} f(\varpi)\, d\varpi \\ &= \tfrac{3}{2(b-a)} f\left(\tfrac{5a+b}{6}\right) - \tfrac{6^{\alpha+1}\Gamma(\alpha+1)}{4(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{5a+b}{6}\right)^-} f(a). \end{aligned} \tag{2.3}$$

Similarly we obtain

$$\begin{aligned}
I_2 &= \tfrac{3}{b-a}\left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right) f\left((1-\varkappa)\tfrac{5a+b}{6}+\varkappa\tfrac{a+b}{2}\right)\Big|_{\varkappa=0}^{\varkappa=1} \\
&\quad -\tfrac{3\alpha}{b-a}\int_0^1 (1-\varkappa)^{\alpha-1} f\left((1-\varkappa)\tfrac{5a+b}{6}+\varkappa\tfrac{a+b}{2}\right) d\varkappa \\
&= \tfrac{9}{8(b-a)} f\left(\tfrac{a+b}{2}\right)+\tfrac{15}{8(b-a)} f\left(\tfrac{5a+b}{6}\right) \\
&\quad -\tfrac{3^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} \left(\tfrac{a+b}{2}-\varpi\right)^{\alpha-1} f(\varpi)\, d\varpi \\
&= \tfrac{9}{8(b-a)} f\left(\tfrac{a+b}{2}\right)+\tfrac{15}{8(b-a)} f\left(\tfrac{5a+b}{6}\right)-\tfrac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{5a+b}{6}\right)^{+}} f\left(\tfrac{a+b}{2}\right), \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
I_3 &= \tfrac{3}{b-a}\left(\tfrac{3}{8}-\varkappa^{\alpha}\right) f\left((1-\varkappa)\tfrac{a+b}{2}+\varkappa\tfrac{a+5b}{6}\right)\Big|_{\varkappa=0}^{\varkappa=1} \\
&\quad +\tfrac{3\alpha}{b-a}\int_0^1 \varkappa^{\alpha-1} f\left((1-\varkappa)\tfrac{a+b}{2}+\varkappa\tfrac{a+5b}{6}\right) d\varkappa \\
&= -\tfrac{15}{8(b-a)} f\left(\tfrac{a+5b}{6}\right)-\tfrac{9}{8(b-a)} f\left(\tfrac{a+b}{2}\right) \\
&\quad +\tfrac{3^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} \left(\varpi-\tfrac{a+b}{2}\right)^{\alpha-1} f(\varpi)\, d\varpi \\
&= -\tfrac{15}{8(b-a)} f\left(\tfrac{a+5b}{6}\right)-\tfrac{9}{8(b-a)} f\left(\tfrac{a+b}{2}\right)+\tfrac{3^{\alpha+1}\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{a+5b}{6}\right)^{-}} f\left(\tfrac{a+b}{2}\right) \qquad (2.5)
\end{aligned}$$

and

$$\begin{aligned}
I_4 &= \tfrac{3}{2(b-a)}(1-\varkappa)^{\alpha} f\left((1-\varkappa)\tfrac{a+5b}{6}+\varkappa b\right)\Big|_{\varkappa=0}^{\varkappa=1} \\
&\quad +\tfrac{3\alpha}{2(b-a)}\int_0^1 (1-\varkappa)^{\alpha-1} f\left((1-\varkappa)\tfrac{a+5b}{6}+\varkappa b\right) d\varkappa \\
&= -\tfrac{3}{2(b-a)} f\left(\tfrac{a+5b}{6}\right)+\tfrac{6^{\alpha+1}\alpha}{4(b-a)^{\alpha+1}}\int_{\frac{a+5b}{6}}^{b} (b-\varpi)^{\alpha-1} f(\varpi)\, d\varpi \\
&= -\tfrac{3}{2(b-a)} f\left(\tfrac{a+5b}{6}\right)+\tfrac{6^{\alpha+1}\Gamma(\alpha+1)}{4(b-a)^{\alpha+1}} I^{\alpha}_{\left(\frac{a+5b}{6}\right)^{+}} f(b). \qquad (2.6)
\end{aligned}$$

Using (2.3)-(2.6) in (2.2), and then multiplying the obtained equality by $\frac{b-a}{9}$, we get the desired result. Thus the proof is completed. □

Definition 2.1. [15] The incomplete beta function is defined for all complex numbers x and y with $\Re(x) > 0$ and $\Re(y) > 0$ and $0 \leq a < 1$ as follows

$$B_a(x, y) = \int_0^a t^{x-1} (1-t)^{y-1} \, dt.$$

Remark 1. *When we set $a = 1$, the incomplete beta function becomes the beta function.*

Definition 2.2. [15] The hypergeometric function is defined for $\Re c > \Re b > 0$ and $|z| < 1$, as follows

$${}_2F_1(a, b, c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-zt)^{-a} \, dt,$$

where $B(.,.)$ is the beta function.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L^1[a, b]$ with $0 \leq a < b$. If $|f'|$ is s-convex in the second sense for some fixed $s \in (0, 1]$, then we have*

$$\begin{aligned}
&\left| \frac{3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)}{8} - \frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \Omega_I^\alpha \left(a, \frac{5a+b}{6}, \frac{a+b}{2}, \frac{a+5b}{6}, b\right) \right| \\
\leq\; & \frac{b-a}{9} \left(\frac{1}{4} B(\alpha+1, s+1) \left(|f'(a)| + |f'(b)| \right) \right. \\
&+ \left(\frac{7(s+1) - 3\alpha}{8(s+1)(\alpha+s+1)} + \frac{2\alpha}{(s+1)(\alpha+s+1)} \left(\frac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}} \right) \left(\left|f'\left(\frac{5a+b}{6}\right)\right| + \left|f'\left(\frac{a+5b}{6}\right)\right| \right) \\
&+ 2 \left(\frac{3}{8(s+1)} - \frac{3}{4(s+1)} \left(1 - \left(\frac{3}{8}\right)^{\frac{1}{\alpha}} \right)^{s+1} + B_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(s+1, \alpha+1) \right. \\
&\left. \left. - B_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(\alpha+1, s+1) \right) \left| f'\left(\frac{a+b}{2}\right) \right| \right),
\end{aligned}$$

where $B(.,.)$ and $B_x(.,.)$ are the beta and the incomplete beta functions respectively.

Proof. From Lemma 2.1, properties of modulus and s-convexity in the second sense of $|f'|$, we have

$$\left| \frac{3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)}{8} - \frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha} \Omega_I^\alpha \left(a, \frac{5a+b}{6}, \frac{a+b}{2}, \frac{a+5b}{6}, b\right) \right|$$

$$\leq \frac{b-a}{9} \left(\int_0^1 \frac{1}{4} \varkappa^\alpha \left| f'\left((1-\varkappa) a + \varkappa \frac{5a+b}{6} \right) \right| d\varkappa \right.$$

$$+\int_0^1 \left|\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right| \left|f'\left((1-\varkappa)\tfrac{5a+b}{6}+\varkappa\tfrac{a+b}{2}\right)\right| d\varkappa$$

$$+\int_0^1 \left|\tfrac{3}{8}-\varkappa^{\alpha}\right| \left|f'\left((1-\varkappa)\tfrac{a+b}{2}+\varkappa\tfrac{a+5b}{6}\right)\right| d\varkappa$$

$$\left.+\int_0^1 \tfrac{1}{4}(1-\varkappa)^{\alpha} \left|f'\left((1-\varkappa)\tfrac{a+5b}{6}+\varkappa b\right)\right| d\varkappa\right)$$

$$\leq \tfrac{b-a}{9}\left(\int_0^1 \tfrac{1}{4}\varkappa^{\alpha}\left((1-\varkappa)^s\left|f'(a)\right|+\varkappa^s\left|f'\left(\tfrac{5a+b}{6}\right)\right|\right)d\varkappa\right.$$

$$+\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)\left((1-\varkappa)^s\left|f'\left(\tfrac{5a+b}{6}\right)\right|+\varkappa^s\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)d\varkappa$$

$$+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)\left((1-\varkappa)^s\left|f'\left(\tfrac{5a+b}{6}\right)\right|+\varkappa^s\left|f'\left(\tfrac{a+b}{2}\right)\right|\right)d\varkappa$$

$$+\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\tfrac{3}{8}-\varkappa^{\alpha}\right)\left((1-\varkappa)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|+\varkappa^s\left|f'\left(\tfrac{a+5b}{6}\right)\right|\right)d\varkappa$$

$$+\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\varkappa^{\alpha}-\tfrac{3}{8}\right)\left((1-\varkappa)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|+\varkappa^s\left|f'\left(\tfrac{a+5b}{6}\right)\right|\right)d\varkappa$$

$$\left.+\int_0^1 \tfrac{1}{4}(1-\varkappa)^{\alpha}\left((1-\varkappa)^s\left|f'\left(\tfrac{a+5b}{6}\right)\right|+\varkappa^s\left|f'(b)\right|\right)d\varkappa\right)$$

$$=\tfrac{b-a}{9}\left(\tfrac{1}{4}\left|f'(a)\right|\int_0^1 \varkappa^{\alpha}(1-\varkappa)^s d\varkappa+\left|f'\left(\tfrac{5a+b}{6}\right)\right|\left(\tfrac{1}{4}\int_0^1 \varkappa^{\alpha+s}d\varkappa\right.\right.$$

$$\left.+\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)(1-\varkappa)^s d\varkappa+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)(1-\varkappa)^s d\varkappa\right)$$

$$+\left|f'\left(\tfrac{a+b}{2}\right)\right|\left(\int\limits_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)\varkappa^{s}d\varkappa+\int\limits_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)\varkappa^{s}d\varkappa\right.$$

$$\left.+\int\limits_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\tfrac{3}{8}-\varkappa^{\alpha}\right)(1-\varkappa)^{s}d\varkappa+\int\limits_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\varkappa^{\alpha}-\tfrac{3}{8}\right)(1-\varkappa)^{s}d\varkappa\right)$$

$$+\left|f'\left(\tfrac{a+5b}{6}\right)\right|\left(\int\limits_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\tfrac{3}{8}-\varkappa^{\alpha}\right)\varkappa^{s}d\varkappa+\int\limits_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\varkappa^{\alpha}-\tfrac{3}{8}\right)\varkappa^{s}d\varkappa\right.$$

$$\left.+\tfrac{1}{4}\int\limits_0^1(1-\varkappa)^{\alpha+s}d\varkappa\right)+\tfrac{1}{4}\left|f'(b)\right|\int\limits_0^1(1-\varkappa)^{\alpha}\varkappa^{s}d\varkappa\Bigg)$$

$$=\tfrac{b-a}{9}\left(\tfrac{1}{4}B(\alpha+1,s+1)\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\right.$$

$$+\left(\tfrac{7(s+1)-3\alpha}{8(s+1)(\alpha+s+1)}+\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left(\left|f'\left(\tfrac{5a+b}{6}\right)\right|+\left|f'\left(\tfrac{a+5b}{6}\right)\right|\right)$$

$$+2\left(\tfrac{3}{8(s+1)}-\tfrac{3}{4(s+1)}\left(1-\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}+B_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)\right.$$

$$\left.\left.-B_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1)\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|\right),$$

where we have used the fact

$$\int\limits_0^1\varkappa^{\alpha}(1-\varkappa)^{s}d\varkappa=\int\limits_0^1(1-\varkappa)^{\alpha}\varkappa^{s}d\varkappa=B(\alpha+1,s+1),\tag{2.7}$$

$$\int\limits_0^1\varkappa^{\alpha+s}d\varkappa=\int\limits_0^1(1-\varkappa)^{\alpha+s}d\varkappa=\tfrac{1}{\alpha+s+1},\tag{2.8}$$

$$\begin{aligned}\int\limits_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)(1-\varkappa)^{s}d\varkappa &= \int\limits_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\varkappa^{\alpha}-\tfrac{3}{8}\right)\varkappa^{s}d\varkappa\\ &= \tfrac{5(s+1)-3\alpha}{8(s+1)(\alpha+s+1)}+\tfrac{\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}},\end{aligned}\tag{2.9}$$

$$(2.10)\qquad \int\limits_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1} \left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)(1-\varkappa)^{s}\,d\varkappa = \int\limits_{0}^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\tfrac{3}{8}-\varkappa^{\alpha}\right)\varkappa^{s}d\varkappa = \tfrac{\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}},$$

$$(2.11)\qquad \int\limits_{0}^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)\varkappa^{s}d\varkappa = \int\limits_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1} \left(\varkappa^{\alpha}-\tfrac{3}{8}\right)(1-\varkappa)^{s}\,d\varkappa = B_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(s+1,\alpha+1)-\tfrac{3}{8(s+1)}\left(1-\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}$$

and

$$(2.12)\qquad \int\limits_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1} \left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)\varkappa^{s}d\varkappa = \int\limits_{0}^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\tfrac{3}{8}-\varkappa^{\alpha}\right)(1-\varkappa)^{s}\,d\varkappa = \tfrac{3}{8(s+1)}-\tfrac{3}{8(s+1)}\left(1-\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}-B_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(\alpha+1,s+1).$$

The proof is completed. □

Corollary 2.1. *In Theorem 2.1, if we take $s=1$, then we get*

$$\begin{aligned}
&\left|\tfrac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\tfrac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Omega_{I}^{\alpha}\left(a,\tfrac{5a+b}{6},\tfrac{a+b}{2},\tfrac{a+5b}{6},b\right)\right| \\
\leq\ & \tfrac{b-a}{9}\left(\tfrac{1}{4(\alpha+1)(\alpha+2)}\left(\left|f'(a)\right|+\left|f'(b)\right|\right)\right. \\
&+\left(\tfrac{14-3\alpha}{16(\alpha+2)}+\tfrac{\alpha}{\alpha+2}\left(\tfrac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}\right)\left(\left|f'\left(\tfrac{5a+b}{6}\right)\right|+\left|f'\left(\tfrac{a+5b}{6}\right)\right|\right) \\
&\left.+\left(\tfrac{16-3(\alpha+1)(\alpha+2)}{8(\alpha+1)(\alpha+2)}+\tfrac{3\alpha}{2(\alpha+1)}\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}}-\tfrac{3\alpha}{4(\alpha+2)}\left(\tfrac{3}{8}\right)^{\frac{2}{\alpha}}\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|\right).
\end{aligned}$$

Corollary 2.2. *In Theorem 2.1, if we take $\alpha = 1$, then we get*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{9}\left(\frac{1}{4(1+s)(s+2)}\left(|f'(a)|+|f'(b)|\right)\right.$$
$$+\left(\frac{7s+4}{8(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+2}\right)\left(\left|f'\left(\frac{5a+b}{6}\right)\right|+\left|f'\left(\frac{a+5b}{6}\right)\right|\right)$$
$$\left.+\left(\frac{3s-2}{4(s+1)(s+2)}+\frac{5}{2(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}\right)\left|f'\left(\frac{a+b}{2}\right)\right|\right).$$

Corollary 2.3. *In Theorem 2.1, if we take $\alpha = s = 1$, then we get*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{25(b-a)}{288}\left(\frac{64|f'(a)|+379\left|f'\left(\frac{5a+b}{6}\right)\right|+314\left|f'\left(\frac{a+b}{2}\right)\right|+379\left|f'\left(\frac{a+5b}{6}\right)\right|+64|f'(b)|}{1200}\right).$$

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1[a,b]$ with $0 \leq a < b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s \in (0,1]$ where $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\left|\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Omega_I^{\alpha}\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\right|$$
$$\leq \frac{b-a}{9}\left(\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+5b}{6}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)\right.$$
$$+\left(\frac{1}{\alpha}\left(\frac{3}{8}\right)^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)+\frac{1}{\alpha}\left(\frac{5}{8}\right)^{p+1}\frac{1}{p+1}\cdot{}_2F_1\left(1-\frac{1}{\alpha},1,p+2,\frac{5}{8}\right)\right)^{\frac{1}{p}}$$
$$\left.\times\left(\left(\frac{\left|f'\left(\frac{5a+b}{6}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+5b}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),$$

where $B(.,.)$ and ${}_2F_1(.,.,.,.)$ are the beta and hypergeometric functions respectively.

Proof. From Lemma 2.1, properties of modulus, Hölder's inequality, and s-convexity in the second sense of $|f'|^q$, we have

$$\left|\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Omega_I^{\alpha}\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\right|$$
$$\leq \frac{b-a}{9}\left(\frac{1}{4}\left(\int_0^1 \varkappa^{\alpha p}d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left((1-\varkappa)a+\varkappa\frac{5a+b}{6}\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1 \left|\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left((1-\varkappa)\tfrac{5a+b}{6}+\varkappa\tfrac{a+b}{2}\right)\right|^q d\varkappa\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|\tfrac{3}{8}-\varkappa^{\alpha}\right|^p d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left((1-\varkappa)\tfrac{a+b}{2}+\varkappa\tfrac{a+5b}{6}\right)\right|^q d\varkappa\right)^{\frac{1}{q}}$$

$$\left.+\tfrac{1}{4}\left(\int_0^1 (1-\varkappa)^{\alpha p} d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left((1-\varkappa)\tfrac{a+5b}{6}+\varkappa b\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\right)$$

$$\leq \tfrac{b-a}{9}\left(\tfrac{1}{4}\left(\int_0^1 \varkappa^{\alpha p} d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1 \left((1-\varkappa)^s\left|f'(a)\right|^q+\varkappa^s\left|f'\left(\tfrac{5a+b}{6}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)^p d\varkappa+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)^p d\varkappa\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left((1-\varkappa)^s\left|f'\left(\tfrac{5a+b}{6}\right)\right|^q+\varkappa^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\tfrac{3}{8}-\varkappa^{\alpha}\right)^p d\varkappa+\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1 \left(\varkappa^{\alpha}-\tfrac{3}{8}\right)^p d\varkappa\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left((1-\varkappa)^s\left|f'\left(\tfrac{a+b}{2}\right)\right|^q+\varkappa^s\left|f'\left(\tfrac{a+5b}{6}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}$$

$$\left.+\tfrac{1}{4}\left(\int_0^1 (1-\varkappa)^{\alpha p} d\varkappa\right)^{\frac{1}{p}}\left(\int_0^1 \left((1-\varkappa)^s\left|f'\left(\tfrac{a+5b}{6}\right)\right|^q+\varkappa^s\left|f'(b)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\right)$$

$$=\tfrac{b-a}{9}\left(\tfrac{1}{4}\left(\tfrac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\tfrac{\left|f'(a)\right|^q+\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+5b}{6}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right.$$

$$+\left(\tfrac{1}{\alpha}\left(\tfrac{3}{8}\right)^{p+\frac{1}{\alpha}}B\left(\tfrac{1}{\alpha},p+1\right)+\tfrac{1}{\alpha}\left(\tfrac{5}{8}\right)^{p+1}\tfrac{1}{p+1}\cdot{}_2F_1\left(1-\tfrac{1}{\alpha},1,p+2,\tfrac{5}{8}\right)\right)^{\frac{1}{p}}$$

$$\left.\times\left(\left(\tfrac{\left|f'\left(\frac{5a+b}{6}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\tfrac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+5b}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right),$$

where we have used the fact that

$$\begin{aligned}
&\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1} \left(\tfrac{3}{8}-(1-\varkappa)^{\alpha}\right)^{p} d\varkappa \\
&= \int_{0}^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left(\tfrac{3}{8}-\varkappa^{\alpha}\right)^{p} d\varkappa = \tfrac{1}{\alpha}\int_{0}^{\frac{3}{8}} \left(\tfrac{3}{8}-u\right)^{p} u^{\frac{1}{\alpha}-1} du \\
&= \tfrac{1}{\alpha}\left(\tfrac{3}{8}\right)^{p+\frac{1}{\alpha}} \int_{0}^{1} z^{\frac{1}{\alpha}-1}(1-z)^{p}\, dz \\
&= \tfrac{1}{\alpha}\left(\tfrac{3}{8}\right)^{p+\frac{1}{\alpha}} B\left(\tfrac{1}{\alpha}, p+1\right)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{0}^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}} \left((1-\varkappa)^{\alpha}-\tfrac{3}{8}\right)^{p} d\varkappa \\
&= \int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1} \left(\varkappa^{\alpha}-\tfrac{3}{8}\right)^{p} d\varkappa = \tfrac{1}{\alpha}\int_{\frac{3}{8}}^{1} \left(u-\tfrac{3}{8}\right)^{p} u^{\frac{1}{\alpha}-1} du \\
&= \tfrac{1}{\alpha}\left(\tfrac{5}{8}\right)^{p+1} \int_{0}^{1} (1-z)^{p}\left(1-\tfrac{5}{8}z\right)^{\frac{1}{\alpha}-1} dz \\
&= \tfrac{1}{\alpha}\left(\tfrac{5}{8}\right)^{p+1} \tfrac{1}{p+1} \cdot {}_2F_1\left(1-\tfrac{1}{\alpha}, 1, p+2, \tfrac{5}{8}\right).
\end{aligned}$$

The proof is completed. □

Corollary 2.4. *In Theorem 2.2, if we take $s=1$, then we get*

$$\begin{aligned}
&\left|\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8} - \frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Omega_I^{\alpha}\left(a, \tfrac{5a+b}{6}, \tfrac{a+b}{2}, \tfrac{a+5b}{6}, b\right)\right| \\
\le\ & \frac{b-a}{9}\left(\frac{1}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|f'(a)|^{q}+\left|f'\left(\frac{5a+b}{6}\right)\right|^{q}}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{a+5b}{6}\right)\right|^{q}+|f'(b)|^{q}}{2}\right)^{\frac{1}{q}}\right)\right. \\
&+\left(\tfrac{1}{\alpha}\left(\tfrac{3}{8}\right)^{p+\frac{1}{\alpha}} B\left(\tfrac{1}{\alpha}, p+1\right) + \tfrac{1}{\alpha}\left(\tfrac{5}{8}\right)^{p+1} \tfrac{1}{p+1} \cdot {}_2F_1\left(1-\tfrac{1}{\alpha}, 1, p+2, \tfrac{5}{8}\right)\right)^{\frac{1}{p}} \\
&\left.\times\left(\left(\frac{\left|f'\left(\frac{5a+b}{6}\right)\right|^{q}+\left|f'\left(\frac{a+b}{2}\right)\right|^{q}}{2}\right)^{\frac{1}{q}} + \left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^{q}+\left|f'\left(\frac{a+5b}{6}\right)\right|^{q}}{2}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Corollary 2.5. *In Theorem 2.2, if we take $\alpha = 1$, then we get*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{72}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(2\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+5b}{6}\right)\right|^q+|f'(b)|^q}{s+1}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{5a+b}{6}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+5b}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.6. *In Theorem 2.2, if we take $\alpha = s = 1$, then we get*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{72}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(2\left(\left(\frac{|f'(a)|^q+\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+5b}{6}\right)\right|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}}\left(\left(\frac{\left|f'\left(\frac{5a+b}{6}\right)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'\left(\frac{a+5b}{6}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b) such that $f' \in L^1[a, b]$ with $0 \leq a < b$. If $|f'|^q$ is s-convex in the second sense for some fixed $s \in (0, 1]$ where $q \geq 1$, then we have*

$$\left|\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Omega_I^{\alpha}\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\right|$$
$$\leq \frac{b-a}{9}\left(\left(\frac{1}{4(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{4}B(\alpha+1,s+1)|f'(a)|^q+\frac{1}{4(\alpha+s+1)}\left|f'\left(\frac{5a+b}{6}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$+\left(\frac{5-3\alpha}{8(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\left(\frac{5(s+1)-3\alpha}{8(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\frac{5a+b}{6}\right)\right|^q\right.\right.$$
$$\left.+\left(\frac{3}{8(s+1)}-\frac{3}{4(s+1)}\left(1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\zeta(s+1,\alpha+1)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{3}{8(s+1)}-\frac{3}{4(s+1)}\left(1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\zeta(s+1,\alpha+1)\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.$$
$$\left.\left.+\left(\frac{5(s+1)-3\alpha}{8(s+1)(\alpha+s+1)}+\frac{2\alpha}{(s+1)(\alpha+s+1)}\left(\frac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\frac{a+5b}{6}\right)\right|^q\right)^{\frac{1}{q}}\right)$$
$$\left.+\left(\frac{1}{4(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\frac{1}{4(\alpha+s+1)}\left|f'\left(\frac{a+5b}{6}\right)\right|^q+\frac{1}{4}B(\alpha+1,s+1)|f'(b)|^q\right)^{\frac{1}{q}}\right),$$

where

$$\zeta(x,y)=B_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(x,y)-B_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}(y,x), \tag{2.13}$$

$B(.,.)$ *and* $B_x(.,.)$ *are beta and incomplete beta functions.*

Proof. From Lemma 2.1, properties of modulus, power mean inequality, and s-convexity in the second sense of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Omega_I^{\alpha}\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\right|\\
\leq\ &\frac{b-a}{9}\left(\left(\int_0^1\frac{1}{4}\varkappa^{\alpha}d\varkappa\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{1}{4}\varkappa^{\alpha}\left|f'\left((1-\varkappa)a+\varkappa\frac{5a+b}{6}\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|\frac{3}{8}-(1-\varkappa)^{\alpha}\right|d\varkappa\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\frac{3}{8}-(1-\varkappa)^{\alpha}\right|\left|f'\left((1-\varkappa)\frac{5a+b}{6}+\varkappa\frac{a+b}{2}\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\\
&+\left(\int_0^1\left|\frac{3}{8}-\varkappa^{\alpha}\right|d\varkappa\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\frac{3}{8}-\varkappa^{\alpha}\right|\left|f'\left((1-\varkappa)\frac{a+b}{2}+\varkappa\frac{a+5b}{6}\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\\
&\left.+\left(\int_0^1\frac{1}{4}(1-\varkappa)^{\alpha}d\varkappa\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{1}{4}(1-\varkappa)^{\alpha}\left|f'\left((1-\varkappa)\frac{a+5b}{6}+\varkappa b\right)\right|^q d\varkappa\right)^{\frac{1}{q}}\right)\\
\leq\ &\frac{b-a}{9}\left(\left(\int_0^1\frac{1}{4}\varkappa^{\alpha}d\varkappa\right)^{1-\frac{1}{q}}\left(\int_0^1\frac{1}{4}\varkappa^{\alpha}\left((1-\varkappa)^s|f'(a)|^q+\varkappa^s\left|f'\left(\frac{5a+b}{6}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left((1-\varkappa)^{\alpha}-\frac{3}{8}\right)d\varkappa+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{3}{8}-(1-\varkappa)^{\alpha}\right)d\varkappa\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\frac{3}{8}-(1-\varkappa)^{\alpha}\right|\left((1-\varkappa)^s\left|f'\left(\frac{5a+b}{6}\right)\right|^q+\varkappa^s\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{3}{8}-\varkappa^{\alpha}\right)d\varkappa+\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\varkappa^{\alpha}-\frac{3}{8}\right)d\varkappa\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\left|\frac{3}{8}-\varkappa^{\alpha}\right|\left((1-\varkappa)^s\left|f'\left(\frac{a+b}{2}\right)\right|^q+\varkappa^s\left|f'\left(\frac{a+5b}{6}\right)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\frac{1}{4}(1-\varkappa)^{\alpha}d\varkappa\right)^{1-\frac{1}{q}}\times\left(\int_0^1\frac{1}{4}(1-\varkappa)^{\alpha}\left((1-\varkappa)^s\left|f'\left(\frac{a+5b}{6}\right)\right|^q+\varkappa^s\left|f'(b)\right|^q\right)d\varkappa\right)^{\frac{1}{q}}\Bigg)$$

$$=\frac{b-a}{9}\left(\left(\frac{1}{4(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left|f'(a)\right|^q\int_0^1\frac{1}{4}\varkappa^{\alpha}(1-\varkappa)^s d\varkappa+\left|f'\left(\frac{5a+b}{6}\right)\right|^q\int_0^1\frac{1}{4}\varkappa^{\alpha+s}d\varkappa\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left((1-\varkappa)^{\alpha}-\frac{3}{8}\right)d\varkappa+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{3}{8}-(1-\varkappa)^{\alpha}\right)d\varkappa\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f'\left(\frac{5a+b}{6}\right)\right|^q\times\left(\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left((1-\varkappa)^{\alpha}-\frac{3}{8}\right)(1-\varkappa)^s d\varkappa+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{3}{8}-(1-\varkappa)^{\alpha}\right)(1-\varkappa)^s d\varkappa\right)\right.$$

$$\left.+\left|f'\left(\frac{a+b}{2}\right)\right|^q\left(\int_0^{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left((1-\varkappa)^{\alpha}-\frac{3}{8}\right)\varkappa^s d\varkappa+\int_{1-\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\frac{3}{8}-(1-\varkappa)^{\alpha}\right)\varkappa^s d\varkappa\right)\right)^{\frac{1}{q}}$$

$$+\left(\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{3}{8}-\varkappa^{\alpha}\right)d\varkappa+\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\varkappa^{\alpha}-\frac{3}{8}\right)d\varkappa\right)^{1-\frac{1}{q}}$$

$$\times\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\left(\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\frac{3}{8}-\varkappa^{\alpha}\right)(1-\varkappa)^s d\varkappa+\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^{1}\left(\varkappa^{\alpha}-\frac{3}{8}\right)(1-\varkappa)^s d\varkappa\right)\right.$$

$$+\left|f'\left(\tfrac{a+5b}{6}\right)\right|^q\left(\int_0^{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}\left(\tfrac{3}{8}-\varkappa^\alpha\right)\varkappa^s d\varkappa+\int_{\left(\frac{3}{8}\right)^{\frac{1}{\alpha}}}^1\left(\varkappa^\alpha-\tfrac{3}{8}\right)\varkappa^s d\varkappa\right)\Bigg)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{4(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left|f'\left(\tfrac{a+5b}{6}\right)\right|^q\int_0^1\tfrac{1}{4}(1-\varkappa)^{\alpha+s}d\varkappa+\left|f'(b)\right|^q\int_0^1\tfrac{1}{4}(1-\varkappa)^\alpha\varkappa^s d\varkappa\right)^{\frac{1}{q}}\Bigg)$$

$$=\tfrac{b-a}{9}\Bigg(\left(\tfrac{1}{4(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{4}B(\alpha+1,s+1)\left|f'(a)\right|^q+\tfrac{1}{4(\alpha+s+1)}\left|f'\left(\tfrac{5a+b}{6}\right)\right|^q\right)^{\frac{1}{q}}$$

$$+\left(\tfrac{5-3\alpha}{8(\alpha+1)}+\tfrac{2\alpha}{\alpha+1}\left(\tfrac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg(\left(\tfrac{5(s+1)-3\alpha}{8(s+1)(\alpha+s+1)}+\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\tfrac{5a+b}{6}\right)\right|^q$$

$$+\left(\tfrac{3}{8(s+1)}-\tfrac{3}{4(s+1)}\left(1-\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\zeta(s+1,\alpha+1)\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\left(\tfrac{5-3\alpha}{8(\alpha+1)}+\tfrac{2\alpha}{\alpha+1}\left(\tfrac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg(\left(\tfrac{3}{8(s+1)}-\tfrac{3}{4(s+1)}\left(1-\left(\tfrac{3}{8}\right)^{\frac{1}{\alpha}}\right)^{s+1}+\zeta(s+1,\alpha+1)\right)\left|f'\left(\tfrac{a+b}{2}\right)\right|^q$$

$$+\left(\tfrac{5(s+1)-3\alpha}{8(s+1)(\alpha+s+1)}+\tfrac{2\alpha}{(s+1)(\alpha+s+1)}\left(\tfrac{3}{8}\right)^{\frac{\alpha+s+1}{\alpha}}\right)\left|f'\left(\tfrac{a+5b}{6}\right)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\left(\tfrac{1}{4(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\tfrac{1}{4(\alpha+s+1)}\left|f'\left(\tfrac{a+5b}{6}\right)\right|^q+\tfrac{1}{4}B(\alpha+1,s+1)\left|f'(b)\right|^q\right)^{\frac{1}{q}}\Bigg),$$

where $\zeta(.,.)$ is defined by (2.13) and we have used (2.7)-(2.12). The proof is achieved.

□

Corollary 2.7. *In Theorem 2.3, if we take $s=1$, then we get*

$$\left|\frac{3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)}{8}-\frac{6^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\Omega_I^\alpha\left(a,\frac{5a+b}{6},\frac{a+b}{2},\frac{a+5b}{6},b\right)\right|$$

$$\leq\frac{b-a}{9}\Bigg(\frac{1}{4(\alpha+1)}\left(\left(\frac{|f'(a)|^q+(\alpha+1)\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{(\alpha+2)}\right)^{\frac{1}{q}}+\left(\frac{(\alpha+1)\left|f'\left(\frac{a+5b}{6}\right)\right|^q+|f'(b)|^q}{\alpha+2}\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{5-3\alpha}{8(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}\right)^{1-\frac{1}{q}}\Bigg(\Bigg(\left(\frac{10-3\alpha}{16(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}\right)\left|f'\left(\frac{5a+b}{6}\right)\right|^q$$

$$+\left(\frac{16-3(\alpha+1)(\alpha+2)}{16(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\left(\frac{16-3(\alpha+1)(\alpha+2)}{16(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{\frac{\alpha+1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q$$

$$+\left(\frac{10-3\alpha}{16(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{\frac{\alpha+2}{\alpha}}\right)\left|f'\left(\frac{a+5b}{6}\right)\right|^q\Bigg)^{\frac{1}{q}}\Bigg)\Bigg).$$

Corollary 2.8. *In Theorem 2.3, if we take $\alpha = 1$, then we get*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{b-a}{9}\left(\frac{1}{8}\left(\left(\frac{2|f'(a)|^q+2(s+1)\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}+\left(\frac{2(s+1)\left|f'\left(\frac{a+5b}{6}\right)\right|^q+2|f'(b)|^q}{(s+1)(s+2)}\right)^{\frac{1}{q}}\right)\right.$$
$$+\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\frac{5s+2}{8(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+2}\right)\left|f'\left(\frac{5a+b}{6}\right)\right|^q\right.\right.$$
$$\left.+\left(\frac{3s-2}{8(s+1)(s+2)}+\frac{5}{4(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\left(\frac{3s-2}{8(s+1)(s+2)}+\frac{5}{4(s+1)(s+2)}\left(\frac{5}{8}\right)^{s+1}\right)\left|f'\left(\frac{a+b}{2}\right)\right|^q\right.$$
$$\left.\left.\left.+\left(\frac{5s+2}{8(s+1)(s+2)}+\frac{2}{(s+1)(s+2)}\left(\frac{3}{8}\right)^{s+2}\right)\left|f'\left(\frac{a+5b}{6}\right)\right|^q\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.9. *In Theorem 2.3, if we take $\alpha = s = 1$, then we get*

$$\left|\frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right)-\frac{1}{b-a}\int_a^b f(u)\,du\right|$$
$$\leq \frac{17(b-a)}{576}\left(\frac{8}{17}\left(\left(\frac{|f'(a)|^q+2\left|f'\left(\frac{5a+b}{6}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{a+5b}{6}\right)\right|^q+|f'(b)|^q}{3}\right)^{\frac{1}{q}}\right)\right.$$
$$\left.+\left(\left(\frac{251\left|f'\left(\frac{5a+b}{6}\right)\right|^q+157\left|f'\left(\frac{a+b}{2}\right)\right|^q}{408}\right)^{\frac{1}{q}}+\left(\frac{157\left|f'\left(\frac{a+b}{2}\right)\right|^q+251\left|f'\left(\frac{a+5b}{6}\right)\right|^q}{408}\right)^{\frac{1}{q}}\right)\right).$$

3. Applications

Let Υ be the partition of the points $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$, and consider the quadrature formula

$$\int_a^b f(u)\,du = \lambda(f,\Upsilon) + R(f,\Upsilon),$$

where

$$\lambda(f,\Upsilon) = \sum_{i=0}^{n-1}\frac{x_{i+1}-x_i}{8}\left(3f\left(\frac{5x_i+x_{i+1}}{6}\right)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+3f\left(\frac{x_i+5x_{i+1}}{6}\right)\right)$$

and $R(f,\Upsilon)$ denotes the associated approximation error.

Proposition 3.1. *Let $n \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $0 \leq a < b$ and $f' \in L^1[a,b]$. If $|f'|$ is a s-convex function, then we have*

$$\begin{aligned} |R(f,\Upsilon)| \leq & \sum_{i=0}^{n-1} \tfrac{(x_{i+1}-x_i)^2}{9} \left(\tfrac{1}{4(1+s)(s+2)} \left(|f'(x_i)| + |f'(x_{i+1})| \right) \right. \\ & + \left(\tfrac{7s+4}{8(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)} \left(\tfrac{3}{8}\right)^{s+2} \right) \left(\left|f'\left(\tfrac{5x_i+x_{i+1}}{6}\right)\right| + \left|f'\left(\tfrac{x_i+5x_{i+1}}{6}\right)\right| \right) \\ & \left. + \left(\tfrac{3s-2}{4(s+1)(s+2)} + \tfrac{5}{2(s+1)(s+2)} \left(\tfrac{5}{8}\right)^{s+1} \right) \left|f'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right| \right). \end{aligned}$$

Proof. Applying Corollary 2.2 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition Υ, we get

$$\begin{aligned} & \left| \tfrac{1}{8} \left(3f\left(\tfrac{5x_i+x_{i+1}}{6}\right) + 2f\left(\tfrac{x_i+x_{i+1}}{2}\right) + 3f\left(\tfrac{x_i+5x_{i+1}}{6}\right) \right) - \tfrac{1}{x_{i+1}-x_i} \int_{x_i}^{x_{i+1}} f(u)\,du \right| \\ \leq & \tfrac{x_{i+1}-x_i}{9} \left(\tfrac{1}{4(1+s)(s+2)} \left(|f'(x_i)| + |f'(x_{i+1})| \right) \right. \\ & + \left(\tfrac{7s+4}{8(s+1)(s+2)} + \tfrac{2}{(s+1)(s+2)} \left(\tfrac{3}{8}\right)^{s+2} \right) \left(\left|f'\left(\tfrac{5x_i+x_{i+1}}{6}\right)\right| + \left|f'\left(\tfrac{x_i+5x_{i+1}}{6}\right)\right| \right) \\ & \left. + \left(\tfrac{3s-2}{4(s+1)(s+2)} + \tfrac{5}{2(s+1)(s+2)} \left(\tfrac{5}{8}\right)^{s+1} \right) \left|f'\left(\tfrac{x_i+x_{i+1}}{2}\right)\right| \right). \end{aligned}$$

Multiplying both sides of above inequality by $(x_{i+1} - x_i)$, and then summing the obtained inequalities for all $i = 0, 1, \ldots, n-1$ and using the triangular inequality, we get the desired result. □

Proposition 3.2. *Let $n \in \mathbb{N}$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) with $0 \leq a < b$ and $f' \in L^1[a,b]$. If $|f'|^q$ is a s-convex function, then we have*

$$\begin{aligned} |R(f,\Upsilon)| \leq & \sum_{i=0}^{n-1} \tfrac{(x_{i+1}-x_i)^2}{72(p+1)^{\frac{1}{p}}} \left(2 \left(\left(\tfrac{|f'(x_i)|^q + \left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q}{s+1} \right)^{\frac{1}{q}} + \left(\tfrac{\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q + |f'(x_{i+1})|^q}{s+1} \right)^{\frac{1}{q}} \right) \right. \\ & \left. + \left(\tfrac{3^{p+1}+5^{p+1}}{8} \right)^{\frac{1}{p}} \times \left(\left(\tfrac{\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q + \left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{s+1} \right)^{\frac{1}{q}} + \left(\tfrac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q + \left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q}{s+1} \right)^{\frac{1}{q}} \right) \right). \end{aligned}$$

Proof. Applying the second inequality of Corollary 2.5 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition Υ, we get

$$\left| \tfrac{1}{8} \left(3f\left(\tfrac{5x_i+x_{i+1}}{6}\right) + 2f\left(\tfrac{x_i+x_{i+1}}{2}\right) + 3f\left(\tfrac{x_i+5x_{i+1}}{6}\right) \right) - \tfrac{1}{x_{i+1}-x_i} \int_{x_i}^{x_{i+1}} f(u)\,du \right|$$

$$\leq \frac{x_{i+1}-x_i}{72(p+1)^{\frac{1}{p}}}\left(2\left(\left(\frac{|f'(x_i)|^q+\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q+|f'(x_{i+1})|^q}{s+1}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}}\times\left(\left(\frac{\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q+\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right)\right).$$

Multiplying both sides of the above inequality by $(x_{i+1}-x_i)$, and then summing the obtained inequalities for all $i=0,1,\ldots,n-1$ and using the triangular inequality, we get the desired result. □

Proposition 3.3. *Let $n\in\mathbb{N}$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) with $0\leq a<b$ and $f'\in L^1[a,b]$. If $|f'|^q$ is a convex function, then we have*

$$|R(f,\Upsilon)|\leq\sum_{i=0}^{n-1}\frac{17(x_{i+1}-x_i)^2}{576}\left(\frac{8}{17}\left(\left(\frac{|f'(x_i)|^q+2\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q+|f'(x_{i+1})|^q}{3}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\left(\frac{251\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q+157\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{408}\right)^{\frac{1}{q}}+\left(\frac{157\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+251\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q}{408}\right)^{\frac{1}{q}}\right)\right).$$

Proof. Applying the second inequality of Corollary 2.9 on the subintervals $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the partition Υ, we get

$$\left|\frac{1}{8}\left(3f\left(\frac{5x_i+x_{i+1}}{6}\right)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+3f\left(\frac{x_i+5x_{i+1}}{6}\right)\right)-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(u)\,du\right|$$

$$\leq\frac{17(x_{i+1}-x_i)}{576}\left(\frac{8}{17}\left(\left(\frac{|f'(x_i)|^q+2\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q+|f'(x_{i+1})|^q}{3}\right)^{\frac{1}{q}}\right)\right.$$

$$\left.+\left(\left(\frac{251\left|f'\left(\frac{5x_i+x_{i+1}}{6}\right)\right|^q+157\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q}{408}\right)^{\frac{1}{q}}+\left(\frac{157\left|f'\left(\frac{x_i+x_{i+1}}{2}\right)\right|^q+251\left|f'\left(\frac{x_i+5x_{i+1}}{6}\right)\right|^q}{408}\right)^{\frac{1}{q}}\right)\right).$$

Multiplying both sides of the above inequality by $(x_{i+1}-x_i)$, and then summing the obtained inequalities for all $i=0,1,\ldots,n-1$ and using the triangular inequality, we get the desired result. □

4. Conclusion

In this study, we have considered the Maclaurin type integral inequalities, which the main results of the paper can be summarized as follows:

(1) A new identity regarding Maclaurin type inequalities is proved.

(2) Some new fractional Maclaurin type inequalities for functions whose first derivatives are s-convex are established.
(3) Some special cases are derived.
(4) Applications of our findings are provided.

Acknowledgement

We would like to thank the editor and the referees for a very careful reading.

References

[1] T. Abdeljawad, A Lyapunov type inequality for fractional operators with nonsingular Mittag-Leffler kernel, *J. Inequal. Appl.*, **130** (2017), 1–11.

[2] T. Abdeljawad, Fractional operators with exponential kernels and a Lyapunov type inequality, *Adv. Difference Equ.*, **313** (2017), 1–11.

[3] M. W. Alomari and S. S. Dragomir, Various error estimations for several Newton-Cotes quadrature formulae in terms of at most first derivative and applications in numerical integration, *Jordan J. Math. Stat*, **7(2)** (2014), 89–108.

[4] N. Azzouza and B. Meftah, Some weighted integral inequalities for differentiable *beta*-convex functions, *J. Interdiscip. Math.*, **25(2)** (2022), 373–393.

[5] D. Baleanu, A. Kashuri, P.O. Mohammed, and B. Meftah, General Raina fractional integral inequalities on coordinates of convex functions, *Adv. Difference Equ.*, **1** (2021), 1–23.

[6] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen, *(German) Publ. Inst. Math.*, **23(37)** (1978), 13–20.

[7] J. Chen and X. Huang, Some new inequalities of Simpson's type for s-convex functions via fractional integrals, *Filomat*, **15(31)** (2017), 4989–4997.

[8] T. Chiheb, B. Meftah and A. Dih, Dual Simpson type inequalities for functions whose absolute value of the first derivatives are preinvex, *Konuralp J. Math.*, **10(1)** (2022), 73–78.

[9] Lj. Dedić, M. Matić and J. Pečarić, On dual Euler-Simpson formulae. *Bull, Belg. Math. Soc. Simon Stevin*, **8(3)** (2001), 479–504.

[10] S. Ghomrani, B. Meftah, W. Kaidouchi and M. Benssaad, Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of the mixed derivatives are co-ordinated $(\log,(\alpha,m))$-preinvex, *Afr. Mat.*, **32(5)** (2021), 925–940.

[11] R. Gorenflo and F. Mainardi, Fractional calculus: integral and differential equations of fractional order, *A. Carpinteri and F. Mainardi (Editors): Fractals and Fractional Calculus in Continuum Mechanics, Springer Verlag, Wien and New York*, (1997), 223–276.

[12] N. Kamouche, S. Ghomrani and B. Meftah, Fractional Simpson like type inequalities for differentiable s-convex functions, *Journal of Applied Mathematics Statistics and Informatics*, **18(1)** (2022), 73–91.

[13] A. Kashuri, B. Meftah and P.O. Mohammed, Some weighted Simpson type inequalities for differentiable s-convex functions and their applications, *J. Frac. Calc. & Nonlinear Sys.*, **1(1)** (2020), 75–94.

[14] A. Kashuri, B. Meftah, P. O. Mohammed, A. A. Lupa, B. Abdalla, Y. S. Hamed and T. Abdeljawad, Fractional weighted Ostrowski type inequalities and their applications, *Symmetry*, **13(968)** (2021), 1–20.

[15] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and applications of fractional differential equations, *North-Holland Math. Studies, 204. Elsevier Science B.V., Amsterdam*, (2006).

[16] B. Meftah, New Ostrowski's inequalities. *Rev. Colombiana Mat.*, **51(1)** (2017), 57–69.

[17] B. Meftah, Fractional Ostrowski type inequalities for functions whose first derivatives are s-preinvex in the second sense, *Int. J. of Anal. and App.*, **15(2)** (2017), 146–154.

[18] B. Meftah, Some new Ostrowski's inequalities for functions whose n^{th} derivatives are logarithmically convex, *Ann. Math. Sil.*, **32(1)** (2017), 275–284.

[19] B. Meftah, Fractional Ostrowski type inequalities for functions whose certain power of modulus of the first derivatives are prequasiinvex via power mean inequality, *J. Appl. Anal.*, **25(1)** (2019), 83–90.

[20] B. Meftah, M. Merad, N. Ouanas and A. Souahi, Some new Hermite-Hadamard type inequalities for functions whose n^{th} derivatives are convex. Acta Comment, *Univ. Tartu. Math.*, **23(2)** (2019), 163–178.

[21] B. Meftah, New integral inequalities Through the φ-preinvexity, *Iran. J. Math. Sci. Inform.*, **15(1)** (2020), 79–83.

[22] B. Meftah and K. Mekalfa, Some weighted trapezoidal inequalities for differentiable log-convex functions, *J. Interdiscip. Math.*, **23** (2020), 1–13.

[23] P. O. Mohammed and T. Abdeljawad, Integral inequalities for a fractional operator of a function with respect to another function with nonsingular kernel, *Adv. Diff. Equ.* **363** (2020), 1–19.

[24] M. A. Noor and M. U. Awan, Some integral inequalities for two kinds of convexities via fractional integrals. Transylv, *J. Math. Mech.*, **5(2)** (2013), 129–136.

[25] J. E. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial orderings, and statistical applications, *Math. in Science and Engineering, 187. Acad. Press, Inc., Boston, MA*, (1992).

[26] J. Pečarić, J. Wen, W.-L. Wang and T. Lu, A generalization of Maclaurin's inequalities and its applications, *Math. Inequal. Appl.*, **8(4)** (2005), 583–598.

[27] G. Rahman, K. S. Nisar, B. Ghanbari and T. Abdeljawad, On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals, *Adv. Diff. Eq.*, **368** (2020), 1–19.

[28] S. Rashid, F. Jarad, M. A. Noor, K. I. Noor, D. Baleanu and J.-B. Liu, . On Grüss inequalities within generalized K-fractional integrals, *Adv. Difference Equ.* **203** (2020), 1–18.

[29] M. Z. Sarikaya, E. Set, H. Yaldiz, and N. Basak, Hermite -Hadamard's inequalities for fractional integrals and related fractional inequalities, *Math. and Comp. Modelling*, **57** (2013), 2403–2407.

(1)Département des Mathématiques, Faculté des mathématiques, de l'informatique et des sciences de la matière, Université 8 mai 1945 Guelma, Algeria.

Email address: (1) djenaoui.saliha@univ-guelma.dz

Email address: (2) badrimeftah@yahoo.fr