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ON WEAKLY k-CLEAN RINGS

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Abstract. In this paper, we offer a new generalization of the k-clean ring that is called weakly k-clean ring. Let $2 \leq k \in \mathbb{N}$. Then the ring R is said to be a weakly k-clean if for each $a \in R$ there exist $u \in U(R)$ and $e \in P_k(R)$ such that $a = u + e$ or $a = u - e$. We obtain some properties of weakly k-clean rings. It is shown that each homomorphic image of a weakly k-clean ring is weakly k-clean. Also, it is proved that the ring $\mathcal{R}[R, S]$ is weakly k-clean if and only if R is k-clean and S is weakly k-clean.

1. Introduction

Let R be an associative ring with identity. The ring R is said to be clean if for each $a \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $a = u + e$ [8]. Clean rings were introduced as a class of exchange rings [8]. The ring R is said to be weakly clean if for each $a \in R$ there exist $u \in U(R)$ and $e \in Id(R)$ such that $a = u + e$ or $a = u - e$ [1, 5, 6, 7]. In [1, Corollary 1.4] it is shown that, an indecomposable weakly clean ring R is either quasilocal or is an indecomposable ring with exactly two maximal ideals in which $2 \in U(R)$. In [5, Theorem 2.1], it is achieved that the ring R is weakly clean if and only if for any $x \in R$, there exists an idempotent $e \in R$ such that $e \in xR$ and $1 - e \in (1 - x)R$ or $1 - e \in (1 + x)R$. In [7, Theorem 8] it is proved that, if R is a commutative ring and $n \geq 2$, then $\mathbb{M}_n(R)$ is weakly clean if and only if R is clean. Let $2 \leq k \in \mathbb{N}$. Then an element $e \in R$ is said to be k-potent if $e^k = e$. Assume that $P_k(R)$ is the set of k-potent elements of ring R. The ring R is said to be k-clean if for each $a \in R$ there exist $u \in U(R)$ and $e \in P_k(R)$ such that $a = u + e$ [9]. In [9,

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Theorem 2.5] it is shown that, if R is a ring and $e \in P_k(R)$ such that the subrings $e^{k-1}Re^{k-1}$ and $(1-e^{k-1})R(1-e^{k-1})$ are k-clean, then R is also k-clean.

In this paper, we introduce the notion of a weakly k-clean ring as a new generalization of a k-clean ring. Let $2 \leq k \in \mathbb{N}$. Then a ring R is said to be a weakly k-clean if for each $a \in R$ there exist $u \in U(R)$ and $e \in P_k(R)$ such that $a = u + e$ or $a = u - e$. We obtain an element-wise characterization of weakly k-clean rings. It is shown that each homomorphic image of a weakly k-clean ring is weakly k-clean (Lemma 2.4). Also, it is proved that the ring $\mathcal{R}[R, S]$ is weakly k-clean if and only if R is k-clean and S is weakly k-clean (Theorem 2.2).

2. Main Results

Let R be a ring and $k \in \mathbb{N}$ such that $k \geq 2$. Then an element $a \in R$ is said to be k-clean if there exist $u \in U(R)$ and $e \in P_k(R)$ of R such that $a = u + e$ [9].

Lemma 2.1. *A ring R is k-clean if and only if for each $a \in R$, there exist $u \in U(R)$ and $e \in P_k(R)$ such that $a = u - e$. That is $R = U(R) + P_k(R)$ if and only if $R = U(R) - P_k(R)$.*

Proof. Assume that $R = U(R) + P_k(R)$, then for each $a \in R$, $-a = u + e$, and so $a = -u - e \in U(R) - P_k(R)$.

Conversely, if $R = U(R) - P_k(R)$, then for each $a \in R$, $-a = u - e$, and so $a = -u + e \in U(R) + P_k(R)$, i.e., R is k-clean. □

Definition 2.1. Let R be a ring and $2 \leq k \in \mathbb{N}$. Then an element $a \in R$ is said to be weakly k-clean if there exist $u \in U(R)$ and $e \in P_k(R)$ of R such that $a = u + e$ or $a = u - e$. A ring R is said to be weakly k-clean if every element of R is weakly k-clean. Every weakly clean ring is weakly k-clean.

It is easy to see from the definition of a weakly k-clean ring that every k-clean ring is weakly k-clean. However, weakly k-clean rings are not k-clean, in general. To see this, we use the following lemma.

Lemma 2.2. *For a commutative ring R, if R is a k-clean ring with $P_k(R) = \{0, 1\}$, then R is local.*

Proof. Suppose that $x \in R$ is a nonunit. Hence it suffices to show that $x \in J(R)$. For $a \in R$, ax is a nonunit. Since R is a k-clean ring with $P_k(R) = \{0, 1\}$, $ax = u + 1$ for some $u \in U(R)$. Hence $1 - ax \in U(R)$. Then $x \in J(R)$, and so R is local. □

In [3, Proposition 16] it was shown that, if R has exactly two maximal ideals and $2 \in U(R)$, then R is an indecomposable weakly clean ring, and so weakly k-clean ring. Thus $\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}$ is weakly k-clean but is not k-clean by Lemma 2.2, since $P_k(\mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}) = \{0, 1\}$.

Lemma 2.3. *Let R be a ring such that $P_k(R) = R$. Then R is k-clean.*

Proof. Let $a \in R$. Then $a - 1 \in P_k(R)$. Since $a = 1 + (a - 1)$, R is k-clean. □

Lemma 2.4. *Let R be a weakly k-clean ring. Then each homomorphic image of R is weakly k-clean.*

Proof. Assume that $h : R \longrightarrow R'$ be a ring homomorphism and R be a weakly k-clean ring. Let $a' \in h(R)$. Then $a' = h(a)$ for some $a \in R$. Since R is weakly k-clean, there exist $u \in U(R)$ and $e \in P_k(R)$ such that $a = u + e$ or $a = u - e$. Since h is a homomorphism, $h(a) = h(u) + h(e)$ or $h(a) = h(u) - h(e)$ and $h(e) = h(e^k) = (h(e))^k$. Hence $h(e) \in P_k(h(R))$. Let u' be the inverse of u in R. Then $h(u)h(u') = h(uu') = h(1) = 1 = h(u'u) = h(u')h(u)$, and so $h(u) \in U(h(R))$. So $h(R)$ is weakly k-clean, as required. □

Lemma 2.5. *Let $k \in \mathbb{N}$ be an odd integer and R be a ring. Then R is a k-clean ring if and only if R is weakly k-clean.*

Proof. Suppose that $k \in \mathbb{N}$ is an odd integer. Hencc $e \in P_k(R)$ if and only if $-e \in P_k(R)$. Then R is a k-clean ring if and only if R is weakly k-clean. □

Theorem 2.1. *Let $\{R_\alpha\}$ be a family of commutative rings. Then the direct product $R = \prod_\alpha R_\alpha$ is weakly k-clean which is not k-clean if and only if each R_α is weakly clean and at most one R_α is not k-clean.*

Proof. Suppose that R is a k-weakly clean ring. Then R_α is a k-weakly clean ring, by Lemma 2.4. Assume that $\alpha_1 \neq \alpha_2$ such that R_{α_1} and R_{α_2}, are not k-clean. Hence there is an $a_{\alpha_1} = u_1 - e_1 \in (U(R_{\alpha_1}) - P_k(R_{\alpha_1})) - (U(R_{\alpha_1}) + P_k(R_{\alpha_1}))$, and there exists $a_{\alpha_2} = u_2 + e_2 \in (U(R_{\alpha_2}) + P_k(R_{\alpha_2})) - (U(R_{\alpha_2}) - P_k(R_{\alpha_2}))$. Define $a = (a_\alpha) \in R$ by

$$a_\alpha = \begin{cases} a_{\alpha_1} & \alpha = \alpha_1 \\ a_{\alpha_2} & \alpha = \alpha_2 \\ 0 & \text{otherwise} \end{cases}$$

Then $a \notin (U(R) + P_k(R)) \cup (U(R) - P_k(R))$, a contradiction.

Conversely, Assume that R_{α_1} is weakly k-clean but not k-clean and that all the other R_α's are clean. Let $a = (a_\alpha) \in R$. If $a_{\alpha_1} = u_{\alpha_1} - e_{\alpha_1} \in U(R_{\alpha_1}) - P_k(R_{\alpha_1})$, then write $a_\alpha = u_\alpha - e_\alpha \in U(R_\alpha) - P_k(R_\alpha)$ for $\alpha \neq \alpha_1$. Thus $a = (u_\alpha) - (e_\alpha) \in U(R) - P(R)$. If $a_{\alpha_1} = u_{\alpha_1} + e_{\alpha_1} \in U(R_{\alpha_1}) + P_k(R_{\alpha_1})$, then write $a_\alpha = u_\alpha + e_\alpha \in U(R_\alpha) + P_k(R_\alpha)$ for $\alpha \neq \alpha_1$. Thus $a = (u_\alpha) + (e_\alpha) \in U(R) + P(R)$. Therefore R is weakly k-clean. □

Let R be a ring and S be a subring of R. Then the set

$$\mathcal{R}[R, S] = \{(a_1, \cdots, a_n, s, s, \cdots) \mid a_i \in R, s \in S, n \geq 1\},$$

with addition and multiplication defined componentwise, is a ring.

Theorem 2.2. *The ring $\mathcal{R}[R, S]$ is weakly k-clean if and only if R is k-clean and S is weakly k-clean.*

Proof. Suppose that $\mathcal{R}[R, S]$ is weakly k-clean. Since $R \oplus R$ is a summand of $\mathcal{R}[R, S]$, and so R is k-clean, by Theorem 2.1. Since S is a homomorphic image of $\mathcal{R}[R, S]$, S is weakly k-clean, by Lemma 2.4.

Conversely, Suppose that $(a_1, \cdots, a_n, s, s, \cdots) \in \mathcal{R}[R, S]$, R is k-clean and S is weakly k-clean. Since $s \in S$, $s = u + e$ or $s = u - e$ for some $u \in U(S)$ and $e \in P_k(S)$. If $s = u + e$, then we write $a_i = u_i + e_i$ where $u_i \in U(R)$ and $e_i \in P_k(R)$ for $1 \leq i \leq n$. Then

$$(a_1, \cdots, a_n, s, s, \cdots) = (u_1, \cdots, u_n, u, u, \cdots) + (e_1, \cdots, e_n, e, e, \cdots),$$

Where $(u_1, \cdots, u_n, u, u, \cdots) \in U(\mathcal{R}[R, S])$ and $(e_1, \cdots, e_n, e, e, \cdots) \in P_k(\mathcal{R}[R, S])$. If $s = u - e$, then we write $a_i = u_i - e_i$ where $u_i \in U(R)$ and $e_i \in P_k(R)$ for $1 \leq i \leq n$. Then

$$(a_1, \cdots, a_n, s, s, \cdots) = (u_1, \cdots, u_n, u, u, \cdots) - (e_1, \cdots, e_n, e, e, \cdots),$$

Where $(u_1, \cdots, u_n, u, u, \cdots) \in U(\mathcal{R}[R,S])$ and $(e_1, \cdots, e_n, e, e, \cdots) \in P_k(\mathcal{R}[R,S])$. Then $\mathcal{R}[R,S]$ is weakly k-clean. □

Example 2.1. (i) *Let $R = \mathcal{R}[\mathbb{Q}, \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)}]$. Then the ring R is weakly k-clean by Theorem 2.2.*

(ii) *Let $M_{\mathbb{N}}(F)$ denote the ring of $\mathbb{N} \times \mathbb{N}$ infinite matrices over a field F in which each column has finitely many nonzero entries and $R_1 = \{A = (a_{ij}) \in M_{\mathbb{N}}(F) \mid \exists\ n_A \in \mathbb{N}, s.t\ \forall i \geq n_A, j \geq 1\ a_{ij} = a_{i+1 j+1}\}$. Consider $T = \{A \in R_1 \mid A^4 = A,\ AB = BA\ \ \forall B \in R_1\}$. Hence T is a weakly 4-clean ring by* [9, Example 2.4]. *Then $R = \mathcal{R}[\mathbb{Q}, T]$ is weakly 4-clean by Theorem 2.2.*

A Morita context is a 6-tuple $\mathcal{M}(R, M, N, S, \phi, \psi)$, where R and S are rings, M is an (R,S)-bimodule, N is a (S,R)-bimodule, and $\phi : M \otimes_S N \longrightarrow R$ and $\psi : N \otimes_R M \longrightarrow S$ are bimodule homomorphisms such that $T(\mathcal{M}) = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is an associative ring with the obvious matrix operations. The ring $T(\mathcal{M})$ is the Morita context ring associated with $\mathcal{M}$. For more on Morita context rings see [2, 4, 10, 11].

Theorem 2.3. *Let R and S be weakly k-clean rings and either R or S is k-clean. Then the Morita context ring $T(\mathcal{M}) = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is weakly k-clean.*

Proof. Suppose that S is k-clean. Let $t = \begin{pmatrix} a & m \\ n & s \end{pmatrix} \in T(\mathcal{M})$. Since R is weakly k-clean, $a = u + e$ or $a = u - e$ for some $u \in U(R)$ and $e \in P_k(R)$. Note $s - nu^{-1}m \in S$. If $a = u + e$, then write $s - nu^{-1}m = v + f$ where $v \in U(S)$ and $f \in P_k(S)$, as S is k-clean. Hence $t = \begin{pmatrix} u & m \\ n & v + nu^{-1}m \end{pmatrix} + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. Since $\begin{pmatrix} 1_R & 0 \\ -nu^{-1} & 1_S \end{pmatrix} \begin{pmatrix} u & m \\ n & v + nu^{-1}m \end{pmatrix} \begin{pmatrix} 1_R & -u^{-1}m \\ 0 & 1_S \end{pmatrix} = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$, $\begin{pmatrix} u & m \\ n & v + nu^{-1}m \end{pmatrix} \in U(T(\mathcal{M}))$. It is clear that $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in P_k(T(\mathcal{M}))$. Then $T(\mathcal{M})$ is weakly k-clean.

If $a = u - e$, then write $s - nu^{-1}m = v - f$ where $v \in U(S)$ and $f \in P_k(S)$, as S

is k-clean. Hence $t = \begin{pmatrix} u & m \\ n & v+nu^{-1}m \end{pmatrix} - \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. As above, $\begin{pmatrix} u & m \\ n & v+nu^{-1}m \end{pmatrix} \in U(T(\mathcal{M}))$ and $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \in P_k(T(\mathcal{M}))$. Then $T(\mathcal{M})$ is weakly k-clean. $\square$

Example 2.2. *Let $R, S = \mathbb{Z}_4$ and $M, N = 2\mathbb{Z}_4$. Since $P_3(\mathbb{Z}_4) = \{0, 1, 3\}$ and $U(\mathbb{Z}_4) = \{1, 3\}$, $\mathbb{Z}_4$ is (weakly) 3-clean. Then the Morita context ring $T(\mathcal{M}) = \begin{pmatrix} R & M \\ N & S \end{pmatrix}$ is weakly 3-clean by Theorem 2.3.*

Here we shall formulate two questions of interest.

Problem 2.1. *When is a matrix ring weakly k-clean?*

Problem 2.2. *Let R be a ring and $e \in P_k(R)$ such that the subrings $e^{k-1}Re^{k-1}$ and $(1-e^{k-1})R(1-e^{k-1})$ are weakly k-clean. Is R also weakly k-clean?*

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References

[1] M.Y. Ahn, D.D. Anderson, Weakly clean rings and almost clean rings, *Rocky Mountain Journal of Mathematics*, **36** (2006), 783–799.

[2] S.A. Amitsur, Rings of quotients and Morita contexts, *Journal of Algebra*, **17(2)** (1971), 273–298.

[3] D.D. Anderson, V.P. Camillo, Commutative rings whose elements are a sum of a unit and an idempotent, *Comm. Algebra*, **30** (2002), 3327–3336.

[4] H. Bass, The Morita Theorems, Mimeographed Notes, Oregon: University of Oregon, (1962).

[5] A.Y.M. Chin, K.T. Qua, A note on weakly clean rings, *Acta Math. Hungra.*, **132** (2001), 113–116.

[6] P.D. Danchev, On weakly clean and weakly exchange rings having the strong property, *Publications de L'Institut Mathematique*, **101** (2017), 135–142.

[7] T. Kosan, S. Sahinkaya, Y. Zhou, On weakly clean rings, *Comm. Algebra*, **45** (2017), 3494–3502.

[8] W.K. Nicholson, Lifting idempotents and exchange rings, *Trans. Amer. Math. Soc.*, **229** (1977), 269–278.

[9] S. Purkait, T.K. Dutta, S. Kar, On m-clean and strongly m-clean rings, *Comm. Algebra*, **48** (2019), 218–227.

[10] A.D. Sands, Radicals and Morita contexts, *J. Algebra*, **24(2)** (1973), 335–345.

[11] G.Tang, C. Li, Y. Zhou, Study of Morita contexts, *Comm. Algebra*, **42(4)** (2014), 1668–1681.

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