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EULER-MARUYAMA APPROXIMATION FOR DIFFUSION PROCESS GENERATED BY DIVERGENCE FORM OPERATOR WITH DISCONTINUOUS COEFFICIENTS

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Abstract. We consider the Euler-Maruyama approximation for time-inhomogeneous one-dimensional stochastic differential equations involving the local time (SDELT), generated by divergence form operators with discontinuous coefficients at zero. We use a space transform in order to remove the local time L_t^0 from the stochastic differential equation of type

$$dX_t = \sigma(t, X_t)dB_t + \tfrac{1}{2}\sigma(t, X_t)\sigma'_x(t, X_t)dt + \beta(t)dL_t^0(X).$$

After that we use a transformation technique that removes the discontinuity from the drift of the new auxiliary equation without local time, such that the coefficients of the obtained time-inhomogeneous SDE are Lipschitz continuous in space. Thus the Euler-Maruyama method can be applied and we provide the rate of strong convergence desired.

1. Introduction

In the last years, various studies have been developed for the numerical simulation of one-dimensional diffusion process X generated by the differential operator

$$A := \frac{\rho}{2}\nabla.\left(a\nabla\right),$$

involving a discontinuous coefficient ρ and a.

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Indeed these operators are of great importance since they appear in a wide range of modelling problems involving diffusion phenomena in discontinuous media: in astrophysics [30], in biology [9], in ecology [8], in geophysics and study of heterogeneous media [11, 22],...

In the case where, $\rho(x) := \dfrac{1}{a(x)}$ and $a(x) := a_+\mathbf{1}_{\{x\geq 0\}} + a_-\mathbf{1}_{\{x<0\}}$, where a_+ and a_- are positive constants, the diffusion process X generated by A is solution to a SDE with local time of type

$$X_t = X_0 + B_t + \beta L_t^0(X), \quad \beta := \frac{a_+ - a_-}{a_+ + a_-}, \tag{1.1}$$

where $(L_t^0(X))_{t\in[0,T]}$ stands for the (symmetric) local time of the unknown process $(X_t)_{t\in[0,T]}$ at point 0.

The solution of equation (1.1) is the well-known process called "skew Brownian motion", which has received great attention in the last years. The reader may find many references concerning study of existence and pathwise uniqueness result of a solution of equation (1.1), and various extensions in the literature. We cite Barlow [1], Benabdallah et al. [4], Étoré and Martinez [13], Harrison and Shepp [18], Lejay [23].

The need for generalizing the concept of skew Brownian motion (SBM) arises from its potential applications in various fields like astrophysics, geophysics, biology, finance and recently to the simulation of diffusion processes with discontinuous coefficients. Indeed the SBM is related essentially to diffusion processes with discontinuous coefficients and to diffusions on graphs. These latter are particularly important in the study of dynamical Hamiltonian systems. In addition, the properties of the SBM become an important tool for solving applied problems by Monte Carlo methods (see [12, 16, 26, 27] among many other works).

The SDE solved by the skew Brownian motion may be generalized as a SDE of type

$$X_t = X_0 + \int_0^t \sigma(X_s)dB_s + \int_{\mathbb{R}} L_t^x(X)\mu(dx), \tag{1.2}$$

where μ is a finite signed measure, that has a mass at the points where ρ or a are discontinuous and $(L_t^x(X))_{t\in[0,T]}$ stands for the (symmetric) local time of the unknown process $(X_t)_{t\in[0,T]}$ at point x.

Numerical analysis for this type of stochastic differential equations has been the subject of many papers from both sides of the theory and application. Recently, many numerical schemes have been proposed and a lot of interesting results have been discovered. One can see Bass and Chen [2], Benabdallah and Bourza [3, 6, 7], Blei and Engelbert [5], Étoré and Martinez [14], Le Gall [21], etc.

This article is an attempt to generalize this family of results in a time-inhomogeneous context. In the case where $\rho(x) = a(x) = \sigma(x)$, we have a look to a time-inhomogeneous version of (1.2), namely

$$\begin{cases} X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s + \frac{1}{2}\int_0^t \sigma(s, X_s)\sigma'_x(s, X_s)ds \\ \qquad + \int_0^t \beta(s)dL_s^0(X), \quad t \in [0, T], \\ X_0 = x_0, \quad x_0 \in \mathbb{R}, \end{cases} \tag{1.3}$$

where $\beta(t) := \dfrac{\sigma(t, 0+) - \sigma(t, 0-)}{\sigma(t, 0+) + \sigma(t, 0-)}$, with $\sigma(t, 0-)$ and $\sigma(t, 0+)$ denote the limits of the function σ on the left and on the right of zero in space respectively, where L_t^0 denotes the local time at 0 for the time t of the semi-martingale X.
One of the possible ways to define it is through the limit

$$L_t^0(X) := \lim_{\varepsilon \longrightarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(|X_s| < \varepsilon)}(X_s)\, d < X >_s, \quad t \geq 0,$$

as it satisfies, for example, $L_t^0(X) := \displaystyle\int_0^t \mathbf{1}_{\left\{X_s = 0\right\}} dL_s^0(X)$, see [[29], Proposition 1.3].

Here the generalization is two fold: first the diffusion coefficient σ and the drift coefficient are now allowed to depend on time, second the function $\beta : [0, T] \longrightarrow (-1, 1)$ is no longer constant but is also allowed to depend on time, where $(L_t^0(X))_{t \in [0,T]}$ stands for the (symmetric) local time at point 0 for the time t of the semi-martingale $X := (X_t)_{t \in [0,T]}$.

In [[15], Theorem 3.5], Étoré and Martinez gave existence and uniqueness result of the solution $X := (X_t)_{t \in [0,T]}$ of SDE of type (1.3) under mild assumptions. The goal of this paper is that under the assumptions that the SDELT (1.3) has a strong solution and that it is unique, we study the conditions under which the Euler-Maruyama scheme $\left\{X_t^n : 0 \leq t \leq T\right\}$ converges strongly to the exact solution $\left\{X_t : 0 \leq t \leq T\right\}$ of the SDELT (1.3).

We face two major problems: the presence of local time in equation (1.3) and the inability to provide a simple discretization scheme, and the presence of a discontinuity for the coefficients appearing in the SDELT (1.3).

In this paper for study the strong convergence for the Euler-Maruyama approximation for the SDELT (1.3), we strongly rely on space transform tricks. First we use a space transform in order to get rid of the local time from the SDELT (1.3). But the auxiliary equation obtained by transforming the original one are shown to have a discontinuous drift. Then, in a second time, we use a transformation technique that removes the discontinuity from the drift, such that the coefficients of the new auxiliary equation are Lipschitz continuous in space. Thus the Euler-Maruyama method can be applied to this transformed SDE. Finally, the approximation can be transformed back, giving an approximation of Euler-Maruyama to the solution of the SDELT (1.3).

Our paper is divided as follows: In Section 2, we present the important assumptions, and a space transforms that can be used to get the result of convergence for this class of stochastic differential equations involving the local time at zero. Our main theorem on the strong convergence of Euler-Maruyama scheme for this class of stochastic differential equations is given in Section 3. Finally, we discuss possible extensions in Section 4.

2. Existence and uniqueness

Let $B := \left(B_t\right)_{0\leq t\leq T}$ be a standard one-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\left(\mathcal{F}_t\right)_{0\leq t\leq T}$ satisfying the usual conditions.

Let $X := \left(X_t\right)_{0\leq t\leq T}$ be a stochastic process involving the local time defined by the stochastic differential equation

$$(2.1)\qquad \begin{cases} X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s + \frac{1}{2}\int_0^t \sigma(s, X_s)\sigma'_x(s, X_s) ds \\ \qquad + \int_0^t \beta(s) dL_s^0(X), \quad t \in [0, T], \\ X_0 = x_0, \end{cases}$$

where $T > 0$ denotes the time horizon, $x_0 \in \mathbb{R}$ is the starting point, we assume that the initial value x_0 is independent of B and satisfies $\mathbb{E}|x_0| < \infty$ and the coefficients σ and β satisfy the following assumptions.

Assumption 2.1. *We assume that the coefficient σ satisfy the following conditions:*

- **(i):** *There exist $\lambda > 0$ and $\Lambda > 0$ such that for any $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\lambda \leq \sigma(t, x) \leq \Lambda.$$

- **(ii):** *The functions $\sigma(t, .)$ and $\sigma'_x(t, .)$ are discontinuous at zero in space, and are class of $C^1(\mathbb{R}^*)$, with a bounded first derivative on $\mathbb{R}^*$.*
- **(iii):** *The limits of the function σ and σ'_x on the left and on the right of 0 exist and are finite.*

Assumption 2.2. *We suppose that there exist $-1 < \underline{m} \leq \overline{m} < 1$ such that for all $t \in [0, T]$, we have $\underline{m} \leq \beta(t) := \dfrac{\sigma(t, 0+) - \sigma(t, 0-)}{\sigma(t, 0+) + \sigma(t, 0-)} \leq \overline{m}$. Moreover, we suppose that β is of class C^1, and that $|\beta'(t)| \leq M$ for any $t \in [0, T]$.*

Notation . *We denote by*

$$\sigma(t, 0-) := \lim_{(t,x) \nearrow (t,0)} \sigma(t, x) \text{ and } \sigma(t, 0+) := \lim_{(t,x) \searrow (t,0)} \sigma(t, x),$$

the respective limits of the function σ on the left and on the right of 0.

Notation . *We denote by*

$$\sigma'_x(t, 0-) := \lim_{(t,x) \nearrow (t,0)} \sigma'_x(t, x) \text{ and } \sigma'_x(t, 0+) := \lim_{(t,x) \searrow (t,0)} \sigma'_x(t, x),$$

the respective limits of the function σ'_x on the left and on the right of 0.

Now, we will introduce a transform space which will be used to get rid of the local time from the SDELT (2.1).

2.1. **Method of removal of the local time.** To construct a practical discretization scheme for the SDELT (2.1) we need to use the following space transform in order to remove the local time from this SDELT.

- We set

$$\beta_t^+ := \frac{\sigma(t, 0+)}{\sigma(t, 0+) + \sigma(t, 0-)} \quad \text{and} \quad \beta_t^- := \frac{\sigma(t, 0-)}{\sigma(t, 0+) + \sigma(t, 0-)}.$$

Let us introduce the $C^{1,2}([0,T]\times\mathbb{R}^*)$ function $r(t,x)$ defined by

$$r(t,x) := \begin{cases} \dfrac{1}{\beta_t^+}x & \text{if } x \geq 0. \\ \\ \dfrac{1}{\beta_t^-}x & \text{if } x < 0. \end{cases}$$

and by $r^{-1}(t,y)$ its inverse map defined by

$$r^{-1}(t,y) := \begin{cases} \beta_t^+ y & \text{if } y \geq 0. \\ \beta_t^- y & \text{if } y < 0. \end{cases}$$

Notation . *We will denote by*

$$b(t,x) = \frac{1}{2}\sigma(t,x)\sigma_x'(t,x),$$

the drift coefficient of the SDELT (2.1).

Proposition 2.1. *Let Assumptions 2.1 and 2.2 be satisfied.*
Then X_t is a solution of equation (2.1), if and only if $Y_t := r(t,X_t)$ is a solution of

$$\begin{cases} Y_t = Y_0 + \int_0^t \bar{\sigma}(s,Y_s)dB_s + \int_0^t \bar{b}(s,Y_s)ds, \quad t\in[0,T], \\ Y_0 = r(0,x_0), \end{cases} \tag{2.2}$$

where

$$\bar{\sigma}(t,y) := \frac{\sigma(t,r^{-1}(t,y))}{(r_y^{-1})'(t,y)} \quad \textit{and} \quad \bar{b}(t,y) := \frac{b(t,r^{-1}(t,y))}{(r_y^{-1})'(t,y)} - \frac{(r_t^{-1})'(t,y)}{(r_y^{-1})'(t,y)},$$

the coefficients of the time-inhomogeneous SDE (2.2).

Proof. In the following, we note $X_t := r^{-1}(t,Y_t)$ with Y_t the unique strong solution of (2.2)(corresponding to the given Brownian motion B). We will show that X_t solves (2.1) . Therefore the result, as X_t is $\mathcal{F}_t$-adapted. Using the change of variable formula proposed by Peskir (see [28]) we get

$$\begin{aligned}
X_t &= r^{-1}(0,Y_0) + \int_0^t (r_t^{-1})'(s,Y_s)ds + \int_0^t (r_y^{-1})'(s,Y_s)dY_s + \frac{1}{2}\int_0^t \beta(s)dL_s^0(Y) \\
&= r^{-1}(0,Y_0) + \int_0^t (r_t^{-1})'(s,Y_s)ds + \int_0^t \sigma(s,r^{-1}(s,Y_s))dB_s + \int_0^t b(s,r^{-1}(s,Y_s))ds \\
&\quad - \int_0^t (r_t^{-1})'(s,Y_s)ds + \frac{1}{2}\int_0^t \beta(s)dL_s^0(Y) \\
&= r^{-1}(0,Y_0) + \int_0^t \sigma(s,r^{-1}(s,Y_s))dB_s + \int_0^t b(s,r^{-1}(s,Y_s))ds + \frac{1}{2}\int_0^t \beta(s)dL_s^0(Y).
\end{aligned}$$

It remains to show that $\frac{1}{2}L_t^0(Y) = L_t^0(X)$.

On one hand the symmetric Tanaka formula gives,

$$\begin{aligned}
|X_t| &= |X_0| + \int_0^t sgn(X_s)dX_s + L_t^0(X) \\
&= |X_0| + \int_0^t sgn(Y_s)\sigma(s,r^{-1}(s,Y_s))dB_s + \int_0^t sgn(Y_s)b(s,r^{-1}(s,Y_s))ds + L_t^0(X),
\end{aligned}$$

where we have used $sgn(X_t) = sgn(Y_t)$ and $sgn(0) = 0$ (for the sign function involved in the symmetric Tanaka formula is the symmetric sign function). On the other hand, using again the formula by Peskir, with the function $(t,y) \mapsto |r^{-1}(t,y)|$, we get

$$\begin{aligned}
|X_t| &= |r^{-1}(t,Y_t)| \\
&= |r^{-1}(0,Y_0)| + \int_0^t sgn(Y_s)(r_t^{-1})'(s,Y_s)ds + \int_0^t sgn(Y_s)\sigma(s,r^{-1}(s,Y_s))dB_s \\
&\quad + \int_0^t sgn(Y_s)b(s,r^{-1}(s,Y_s))ds - \int_0^t sgn(Y_s)(r_t^{-1})'(s,Y_s)ds + \frac{1}{2}L_t^0(Y) \\
&= |r^{-1}(0,Y_0)| + \int_0^t sgn(Y_s)\sigma(s,r^{-1}(s,Y_s))dB_s + \int_0^t sgn(Y_s)b(s,r^{-1}(s,Y_s))ds \\
&\quad + \frac{1}{2}L_t^0(Y).
\end{aligned}$$

By unicity of the decomposition of a semi-martingale we get $\frac{1}{2}L_t^0(Y) = L_t^0(X)$ and we are done.

One can obtain in a similar fashion as above that starting from the unique strong solution X of equation (2.1), the process $Y_t := r(t,X_t)$ is a solution of equation (2.2). □

The following lemma shows the properties of the coefficients of SDE (2.2).

Lemma 2.1. *Let Assumptions 2.1 and 2.2 hold. Then*

(i): *There exist $\lambda > 0$ and $\Lambda > 0$ such that for any $(t, x) \in [0, T] \times \mathbb{R}$,*

$$\lambda \leq \bar{\sigma}(t, x) \leq \Lambda.$$

(ii): *$\bar{\sigma}$ is Lipschitz continuous in space.*

(iii): *$\bar{b}$ is piecewise Lipschitz in space, with a discontinuity at zero in space.*

Proof.

(ii): Define $\tilde{\sigma}(t, y) := \dfrac{\sigma(t, y)}{(r_y^{-1})'(t, y)}$.

$\tilde{\sigma}$ is Lipschitz continuous in space on each of the open intervals $(-\infty, 0)$ and $(0, +\infty)$. From the continuity of $\tilde{\sigma}$ in space we conclude that $\tilde{\sigma}$ is Lipschitz continuous in space on each of the closed intervals $(-\infty, 0]$ and $[0, +\infty)$ (see Lemma 2.2 in [24]).

Let L_1, L_2 denote the respective Lipschitz constants and let $L := max(L_1, L_2)$. Now let $x, y \in \mathbb{R}$ and without loss of generality. assume that $y < x$. If x and y are in the same interval, then it is obvious that $|\tilde{\sigma}(t, x) - \tilde{\sigma}(t, y)| \leq L|x - y|$. Otherwise, we have $y \leq 0 \leq x$.

$$\begin{aligned} |\tilde{\sigma}(t, x) - \tilde{\sigma}(t, y)| &= |\tilde{\sigma}(t, x) - \tilde{\sigma}(t, 0) + \tilde{\sigma}(t, 0) - \tilde{\sigma}(t, y)| \\ &\leq |\tilde{\sigma}(t, x) - \tilde{\sigma}(t, 0)| + |\tilde{\sigma}(t, 0) - \tilde{\sigma}(t, y)| \\ &\leq L|x - 0| + L|0 - y| \\ &\leq L|x - y| + L|x - y| \\ &\leq 2L|x - y|. \end{aligned}$$

Since $r^{-1}(t, y)$ is Lipschitz in space. Thus, $\bar{\sigma}$ is Lipschitz continuous in space.

(iii): Define $\tilde{b}(t, y) := \dfrac{b(t, y)}{(r_y^{-1})'(t, y)} - \dfrac{(r_t^{-1})'(t, y)}{(r_y^{-1})'(t, y)}$.

$(r_y^{-1})'(t, y)$ is piecewise C^1 with respect to y, measurable with respect to (t, y), uniformly positive, and bounded.

Since $(r_t^{-1})'(t, y)$ is Lipschitz in space, and $(r_t^{-1})'(t, 0) = 0$, then $\dfrac{(r_t^{-1})'(t, y)}{(r_y^{-1})'(t, y)}$ is piecewise Lipschitz and continuous on $\mathbb{R}$ respect to y. Thus $\dfrac{(r_t^{-1})'(t, y)}{(r_y^{-1})'(t, y)}$ is Lipschitz on $\mathbb{R}$ respect to y by Lemma 2.2 in [24].

On the other hand, b is piecewise Lipschitz in space on each of the open intervals $(-\infty, 0)$ and $(0, +\infty)$. Thus, $\tilde{b}$ is piecewise Lipschitz in space on each of the open intervals $(-\infty, 0)$ and $(0, +\infty)$.
Since $r^{-1}(t, y)$ is Lipschitz in space. Hence $\bar{b}$ is piecewise Lipschitz in space, with a discontinuity at zero in space, by the fact that the composition of Lipschitz functions is Lipschitz.

□

Inspired by [25], we will use the transform G to get a new auxiliary equation with Lipschitz coefficients in space.

2.2. **Construction of the transform** G**.** To prove existence and uniqueness of a strong solution to (2.2) and construct a practical discretization scheme for the SDE (2.2), we are going to construct a transform $G : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that the process defined by $Z_t := G(t, Y_t)$ satisfies a SDE with Lipschitz coefficients, and therefore has a solution by Itô's classical theorem on existence and uniqueness of solutions [19].

For this define the following bump function on $\mathbb{R}$, which we need to localize the impact of the transform G

$$\varphi(u) = \begin{cases} (1-u)^3(1+u)^3 & \text{if } |u| \leq 1. \\ 0 & \text{else .} \end{cases}$$

The function φ has the following properties

(1) φ defines a C^2 function on all of $\mathbb{R}$.
(2) $\varphi(0) = 1$, $\varphi'(0) = 0$, $\varphi''(0) = -6$.
(3) $\varphi(\pm 1) = \varphi'(\pm 1) = \varphi''(\pm 1) = 0$.

We define the transform $G : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$G(t, y) = y + \alpha(t)\bar{\varphi}(y), \ (t, y) \in [0.T] \times \mathbb{R},$$

where $\bar{\varphi}(y) = \varphi(\frac{y}{c})y|y|$, with $c > 0$ is constant and the function $\alpha : [0, T] \to \mathbb{R}^*$ is of class C^1, and that $|\alpha'(t)| \leq M \ \forall t \in [0, T]$.

Now, we are going to show that the constant c can be chosen in a way such that $G(t, .)$ has a global inverse $\forall t \in [0, T]$.

Lemma 2.2. *Let* $c < \frac{1}{6\|\alpha\|_\infty}$.
Then $G'_x(t,x) > 0$, $\forall (t,x) \in [0,T]\times\mathbb{R}$. *Furthermore,* $G'_x(t,x) = 1$, *for all* $|x| > c$ *and* $t \in [0,T]$.
Therefore $G(t,.)$ *has a global inverse* $[G(t,.)]^{-1}$ *for any* $t \in [0,T]$.

Proof. Differentiating G yields,

$$G'_x(t,x) = 1 - \frac{6\alpha(t)x^2|x|}{c^2}\left(1+\frac{x}{c}\right)^2\left(1-\frac{x}{c}\right)^2 + 2\alpha(t)|x|\left(1+\frac{x}{c}\right)^3\left(1-\frac{x}{c}\right)^3.$$

Then, by studying both positive and negative case of the function α, we conclude that the sufficient condition for $G'_x(t,x)$ to be positive is $c < \frac{1}{6\|\alpha\|_\infty}$. □

Without loss of generality, we always choose $c < \frac{1}{6\sup_{t\in[0,T]}|\alpha(t)|}$, such that $G(t,.)$ has a global inverse.

Remark 1. *Although* $[G(t,.)]^{-1} \notin C^2(\mathbb{R})$, *Itô's formula holds for* $[G(t,.)]^{-1}$ *for any* $t \in [0,T]$, *see (*[20]*, 5 . Problem 7.3).*

Before applying the transformation G to SDE (2.2), we state the following properties satisfied by G

(1) $G(t,.)$ has a global inverse for any $t \in [0,T]$.
(2) $G'_x(t,x) > 0, \quad \forall (t,x) \in [0,T]\times\mathbb{R}$.
(3) G and G^{-1} are Lipschitz continuous in space.
(4) Itô's formula holds for $[G(t,.)]^{-1}$ for any $t \in [0,T]$.
(5) G'_x is a Lipschitz continuous function in space.

Next, we going to show that the coefficients of the new time-inhomogeneous SDE obtained by the transform G are Lipschitz in space. Formally define $Z_t := G(t,Y_t)$, we have

$$\begin{aligned} dZ_t &= dY_t + \alpha(t)\bar{\varphi}'(Y_t)dY_t + \alpha'(t)\bar{\varphi}(Y_t)dt + \frac{1}{2}\alpha(t)\bar{\varphi}''(Y_t)d < Y >_t \\ &= \left(\bar{b}(t,Y_t) + \alpha(t)\bar{\varphi}'(Y_t)\bar{b}(t,Y_t) + \frac{1}{2}\alpha(t)\bar{\varphi}''(Y_t)\bar{\sigma}^2(t,Y_t) + \alpha'(t)\bar{\varphi}(Y_t)\right)dt \\ &\quad + \left(\bar{\sigma}(t,Y_t) + \alpha(t)\bar{\varphi}'(Y_t)\bar{\sigma}(t,Y_t)\right)dB_t \\ &= \hat{b}(t,Z_t)dt + \hat{\sigma}(t,Z_t)dB_t, \qquad (2.3) \end{aligned}$$

where

$$\begin{aligned}\hat{b}(t,z) &:= \bar{b}(t,G^{-1}(t,z))+\alpha(t)\bar{\varphi}'(G^{-1}(t,z))\bar{b}(t,G^{-1}(t,z))\\ &\quad+\frac{1}{2}\alpha(t)\bar{\varphi}''(G^{-1}(t,z))\bar{\sigma}^2(t,G^{-1}(t,z))+\alpha'(t)\bar{\varphi}(G^{-1}(t,z)),\\ \hat{\sigma}(t,z) &:= \bar{\sigma}(t,G^{-1}(t,z))+\alpha(t)\bar{\varphi}'(G^{-1}(t,z))\bar{\sigma}(t,G^{-1}(t,z)).\end{aligned}$$

Proposition 2.2. *Let Lemma 2.1 be satisfied.*

Then the time inhomogeneous SDE for $G(t,Y_t)$ has Lipschitz coefficients in space.

Proof. The transformed diffusion coefficient is given by

$$\hat{\sigma}(t,z)=G'_z(t,G^{-1}(t,z))\bar{\sigma}(t,G^{-1}(t,z)).$$

Since G^{-1}, G'_z and $\bar{\sigma}$ are Lipschitz in space, the mappings $z\mapsto G'_z(t,G^{-1}(t,z))$ and $z\mapsto\bar{\sigma}(t,G^{-1}(t,z))$ are both Lipschitz in space and bounded, thus their product $\hat{\sigma}(t,z)$ is Lipschitz in space.

Due to Lemma 2.1.(iii), and $\lim_{(t,h)\to(t,0)}\bar{\varphi}'(t,h)=0$, we can see that the mapping $z\mapsto\bar{\varphi}'(t,G^{-1}(t,z))\bar{b}(t,G^{-1}(t,z))$ is Lipschitz in space from (Lemma 2.4, [24]).

Since G^{-1}, $\bar{\varphi}$ are Lipschitz in space, we see that the mapping $z\mapsto\alpha'(t)\bar{\varphi}(G^{-1}(t,z))$ is Lipschitz in space.

To show that $\hat{b}$ is lipschitz in space, we need to choose the function α such that the mapping

$z\mapsto\bar{b}(t,G^{-1}(t,z))+\frac{1}{2}\alpha(t)\bar{\varphi}''(G^{-1}(t,z))\bar{\sigma}^2(t,G^{-1}(t,z))$ is continous in space. i.e.,

$$\bar{b}(t,0+)+\frac{1}{2}\alpha(t)\bar{\varphi}''(0+)\bar{\sigma}^2(t,0)=\bar{b}(t,0-)+\frac{1}{2}\alpha(t)\bar{\varphi}''(0-)\bar{\sigma}^2(t,0),$$

Thus we get, for the choice

$$\begin{aligned}\alpha(t) &= -2\frac{\bar{b}(t,0+)-\bar{b}(t,0-)}{\left(\bar{\varphi}''(0+)-\bar{\varphi}''(0-)\right)\bar{\sigma}^2(t,0)}\\ &= \frac{\bar{b}(t,0-)-\bar{b}(t,0+)}{2\bar{\sigma}^2(t,0)}\\ &= \frac{1}{4}\frac{\sigma'_x(t,0-)-\sigma'_x(t,0+)}{\sigma(t,0+)+\sigma(t,0-)}.\end{aligned}$$

that $\hat{b}$ is continous in space.

Since $\bar{\varphi}''$ vanishes outside $[-c,c]$, the mapping

$z\mapsto\frac{1}{2}\alpha(t)\bar{\varphi}''(G^{-1}(t,z))\bar{\sigma}^2(t,G^{-1}(t,z))+\bar{b}(t,G^{-1}(t,z))$ is piecewise Lipschitz in space.

Thus $\hat{b}$ is piecewise Lipschitz in space, and continuous in space with the appropriate choice of the function α, then it is Lipschitz in space as well by [24] Lemma 2.4. Altogether we have that the SDE (2.3) for Z has Lipschitz coefficient $\hat{b}$ and $\hat{\sigma}$. □

Finally, we are ready to prove existence and uniqueness of a solution to the one-dimensional SDE (2.2).

Proposition 2.3. *Let Lemma 2.1 be satisfied, i.e., $\bar{b}$ is piecewise Lipschitz in space with at most one jump at 0, $\bar{\sigma}$ is Lipschitz in space and does not vanish in the discontinuity of $\bar{b}$, and G is globally invertible.*
Then the one-dimensional SDE (2.2) has a unique global strong solution.

Proof. Since the SDE (2.3) has Lipschitz coefficients by Proposition 2.2, it follows that (2.3) has a unique global strong solution for the initial value $G(0, Y_0)$.
Furthermore, the transformation G has a global inverse G^{-1} by Lemma 2.2, which inherits the smoothness from G. Itô's formula holds for G^{-1} see Remark 1.
Applying Its formula to G^{-1}, we obtain that $G^{-1}(t, Z)$ satisfies

$$dY_t = \bar{\sigma}(t, Y_t)dB_t + \bar{b}(t, Y_t)dt, \quad Y_0 = y_0.$$

Setting $Y_t := G^{-1}(t, Z_t)$ closes the proof. □

3. The main result

In this section, we provide the description of the used method and the convergent result.

3.1. **Euler-Maruyama approximation.** For each $n \geq 1$, we define the functions $\eta_n : [0, T] \longrightarrow [0, T]$ by $\eta_n(T) := \frac{n-1}{n}T$ and $\eta_n(s) = \frac{kT}{n}$, if $s \in [\frac{kT}{n}, \frac{(k+1)}{n}T]$, for $k = 0, 1, ..., n - 1$.

We now define the continuous Euler-Maruyama scheme for the second auxiliary SDE (2.3), as the solutions of

$$\begin{cases} Z_t^n = Z_0^n + \int_0^t \hat{\sigma}(\eta_n(s), Z_{\eta_n(s)}^n)dB_s + \int_0^t \hat{b}(\eta_n(s), Z_{\eta_n(s)}^n)ds, \\ Z_0^n = Z_0, \end{cases} \tag{3.1}$$

for each $n \geq 1$.

To study the convergence between Z_t and Z_t^n, where Z_t is defined by (2.3) and Z_t^n is defined by (3.1), we will make use of the following assumption.

Assumption 3.1. *We assume that the coefficients* $\hat{b}$ *and* $\hat{\sigma}$ *are* α*-H¨lder continous with* $\alpha \in (0,1]$ *in time i.e,* $\exists\, K > 0$*, such that* $\forall z \in \mathbb{R}$, $\forall (s,t) \in [0,T]$,

$$|\hat{b}(t,z) - \hat{b}(s,z)| + |\hat{\sigma}(t,z) - \hat{\sigma}(s,z)| \leq K(1+|z|)|t-s|^{\alpha}. \tag{3.2}$$

3.2. **Description of the used method.** Introduced in [4], we present the method of strong convergence of Euler-Maruyama scheme for one-dimensional stochastic differential equations involving the local time of the unknown process by the following steps:

(i) We use the transformation $y = r(t,x)$ (see Proposition 2.1) in order to remove the local time L_t^0 from the stochastic differential equation (2.1).

(ii) We use the transformation $z = G(t,y)$ in order to remove the discontinuity frome the drift of the auxiliary SDE (2.2).

(iii) We study the convergence of Euler-Maruyama scheme of equation (2.3).

(iv) We use the transformation inverse $y = G^{-1}(t,z)$ in order to obtain the formulas Y_t and Y_t^n as follows
   (a) $Y_t = G^{-1}(t, Z_t)$, where Z_t is given by (2.3).
   (b) $Y_t^n = G^{-1}(t, Z_t^n)$, where Z_t^n is given by (3.1).

(iv) We use the transformation inverse $x = r^{-1}(t,y)$ in order to obtain the formulas X_t and X_t^n as follows
   (a) $X_t = r^{-1}(t, Y_t)$.
   (b) $X_t^n = r^{-1}(t, Y_t^n)$.

(vi) We conclude the rate of strong convergence between the solution by Euler-Maruyama scheme X_t^n and the solution X_t for the stochastic differential equation (2.1).

3.3. **Main theorem.** Let X_t be defined as in equation (2.1). Let X_t^n be the Euler-Maruyama scheme for equation (2.1). Here, we suppose that Proposition 2.3 is satisfied and the constant C, which does not depend on n, may change from line to line.

Theorem 3.1. *Let Assumption 3.1 holds. Then for any $p > 0$, there exists a constant $C > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t - X_t^n|^p\right] \leq \begin{cases} \dfrac{C}{n^{\frac{p}{2}}} & , \text{if } \alpha \in [\frac{1}{2}, 1]. \\ \\ \dfrac{C}{n^{\alpha p}} & , \text{ if } \alpha \in (0, \frac{1}{2}). \end{cases}$$

3.4. **Proof of Theorem 3.1.** Before going on, let us recall two useful standard estimation (see [17]).

Lemma 3.1. *Let $p \geq 1$ There exists a constant C such that*

$$\mathbb{E}[\sup_{t\in[0,T]}|Z_t|^p] \leq C(1 + |z_0|^p). \tag{3.3}$$

As an immediate consequence we get the following lemma.

Lemma 3.2. *Let $p \geq 1$ There exists a constant C such that for all $0 \leq s \leq t \leq T$. We have*

$$\mathbb{E}[|Z_t - Z_s|^p] \leq C(1 + |z_0|^p)(t - s)^{\frac{p}{2}}. \tag{3.4}$$

Proof of Theorem 3.1: According to Proposition 2.3, X_t is a unique solution of SDELT (2.1) and X_t^n is a solution by Euler-Maruyama scheme of SDE (2.1), then

$$\begin{aligned} \mathbb{E}[\sup_{s\in[0,T]}|X_s - X_s^n|^p] &= \mathbb{E}[\sup_{s\in[0,T]}|r^{-1}(s, Y_s) - r^{-1}(s, Y_s^n)|^p] \\ &\leq |L_r|^p\, \mathbb{E}[\sup_{s\in[0,T]}|Y_s - Y_s^n|^p] \\ &= |L_r|^p\, \mathbb{E}[\sup_{s\in[0,T]}|G^{-1}(s, Z_s) - G^{-1}(s, Z_s^n)|^p] \\ &\leq |K|^p\, \mathbb{E}[\sup_{s\in[0,T]}|Z_s - Z_s^n|^p]. \end{aligned} \tag{3.5}$$

We estimate $\mathbb{E}[\sup_{s\in[0,T]}|Z_s - Z_s^n|^p]$ for all $s \in [0, T]$ and $p > 0$.

By using Jensen's inequality, the case $0 < p < 2$ is obtained from the case $p \geq 2$, so we assume that $p \geq 2$.

Applying the inequality

$$|\sum_{i=1}^{\ell} a_i|^q \leq \ell^{q-1}\sum_{i=1}^{\ell}|a_i|^q, \tag{3.6}$$

for any $\ell \in \mathbb{N}, a_i \in \mathbb{R}$ and $q \geq 1$. There exists a positive constant C such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq s\leq T}|Z_s - Z_s^n|^p\right] &\leq 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})dB_s\right|^p\right] \\
&\quad + 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \hat{b}(s,Z_s) - \hat{b}(\eta_n(s), Z^n_{\eta_n(s)})ds\right|^p\right] \\
&\leq 2^{p-1}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s) Z^n_{\eta_n(s)})dB_s\right|^p\right] \\
&\quad + C\int_0^T \mathbb{E}\left[\left|\hat{b}(s,Z_s) - \hat{b}(\eta_n(s), Z^n_{\eta_n(s)})\right|^p\right] ds. \qquad (3.7)
\end{aligned}$$

It follows from Burkholder-Davis-Gundy's inequality that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})dB_s\right|^p\right] \\
\leq\ & C\mathbb{E}\left[\left\{\int_0^T |\hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})|^2 ds\right\}^{\frac{p}{2}}\right] \\
\leq\ & C\,\mathbb{E}\left[\int_0^T |\hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})|^p ds\right].
\end{aligned}$$

Since $\hat{\sigma}$ is Lipschitz and by using (3.2) and the inequality (3.6) we get

$$\begin{aligned}
\left|\hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})\right|^p &\leq C\Big\{\left|\hat{\sigma}(s,Z_s) - \hat{\sigma}(s, Z_{\eta_n(s)})\right|^p \\
&\quad + \left|\hat{\sigma}(s,Z_{\eta_n(s)}) - \hat{\sigma}(\eta_n(s), Z_{\eta_n(s)})\right|^p \\
&\quad + \left|\hat{\sigma}(\eta_n(s),Z_{\eta_n(s)}) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})\right|^p\Big\} \\
&\leq C(\left|Z_s - Z_{\eta_n(s)}\right|^p + (1+|Z_{\eta_n(s)}|^p)\,|s-\eta_n(s)|^{\alpha p} \\
&\quad + \left|Z_{\eta_n(s)} - Z^n_{\eta_n(s)}\right|^p).
\end{aligned}$$

By (3.3),(3.4) and $0 \leq s - \eta_n(s) \leq \frac{T}{n}$ we get

$$\begin{aligned}
&\mathbb{E}\left[\left|\hat{\sigma}(s,Z_s) - \hat{\sigma}(\eta_n(s), Z^n_{\eta_n(s)})\right|^p\right] \\
\leq\ & C\left(\frac{1+|z_0|^p}{n^{\frac{p}{2}}} + \frac{1+|z_0|^p}{n^{\alpha p}} + \mathbb{E}\left[\left|Z_{\eta_n(s)} - Z^n_{\eta_n(s)}\right|^p\right]\right). \qquad (3.8)
\end{aligned}$$

This inequality stay true when we change $\hat{\sigma}$ by $\hat{b}$ in left member, and by (3.8) we have

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|Z_s - Z_s^n|^p\right] \leq C\left(\frac{1+|z_0|^p}{n^{(\alpha\wedge\frac{1}{2})p}} + \int_0^T \mathbb{E}\left[\sup_{0\leq u\leq s}\left|Z_{\eta_n(u)} - Z^n_{\eta_n(u)}\right|^p ds\right]\right).$$

From Gronwall's inequality we get

$$\mathbb{E}\left[\sup_{0\le s\le T}|Z_s - Z_s^n|^p\right] \le \frac{C}{n^{(\alpha\wedge\frac{1}{2})p}}. \tag{3.9}$$

Then we consider the following two cases

Case 1: $\alpha \in [\frac{1}{2}, 1]$. From (3.9) we obtain

$$\mathbb{E}\left[\sup_{0\le s\le T}|Z_s - Z_s^n|^p\right] \le \frac{C}{n^{\frac{p}{2}}}. \tag{3.10}$$

From (3.5) and (3.10) we get

$$\mathbb{E}\left[\sup_{0\le s\le T}|X_s - X_s^n|^p\right] \le \frac{C}{n^{\frac{p}{2}}}.$$

Case 2: $\alpha \in (0, \frac{1}{2})$. From (3.9)we obtain

$$\mathbb{E}\left[\sup_{0\le s\le T}|Z_s - Z_s^n|^p\right] \le \frac{C}{n^{\alpha p}}. \tag{3.11}$$

From (3.5) and (3.11) we get

$$\mathbb{E}\left[\sup_{0\le s\le T}|X_s - X_s^n|^p\right] \le \frac{C}{n^{\alpha p}}.$$

Hence the proof of the Theorem 3.1 is complete. □

4. Conclusion

The basic idea of the method presented here depends highly on the space transform tricks, such that no local time appears in the new auxiliary equation and the coefficients are Lipschitz continuous in space.

It would be interesting to see, if the result of Theorem 3.1 remains true in the general case, where the discontinuity of the drift coefficient b, the diffusion coefficient σ, and the local time appear at distinct space points or follow time-curves.

Conflict of interest

The author declares there is no conflict of interest.

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