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MODERN FUNCTIONAL ANALYSIS IN THE THEORY OF SEQUENCE SPACES AND MATRIX TRANSFORMATIONS

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Abstract. Many concepts and theories in functional analysis have turned out to be powerful and widely used tools in operator theory, in particular in the theory of matrix transformations between sequences spaces in summability. We give an introduction to the basic theory of FK, BK, AK and AD spaces, the various types of dual spaces of sequence spaces, and apply the general results to the characterisations of classes of matrix operators between certain sequence spaces that arise in summability. We also study the Hausdorff measure of noncompactness and its applications to the characterisations of compact operators between sequence spaces.

This is a survey paper which also includes some results of the author's joint research with V. Rakočević and I. Djolević at the Department of Mathematics of the Faculty of Science and Mathematics at the University of Niš, Serbia. Although many of the results are probably known to specialists, the proofs are included for the convenience of those readers who may not be too familiar with the subject, and an appendix is added at the end containing the fundamental theorems in functional analysis in the versions they are applied.

1. Introduction, Standard Notations and Well–Known Results

Let X be a normed space. Then we denote the open unit ball and the unit sphere in X by $B_X = \{x \in X : \|x\| < 1\}$ and $S_X = \{x \in X : \|x\| = 1\}$.

Let X and Y be Banach spaces. Then $\mathcal{B}(X,Y)$ denotes the set of all bounded linear operators $L : X \to Y$; $\mathcal{B}(X,Y)$ is a Banach space with the operator norm defined by $\|L\| = \sup_{x\in S_X} \|L(x)\|$ for all $L \in \mathcal{B}(X,Y)$.

Let X be a linear metric space. A Schauder basis of X is a sequence $(b_n)_{n=0}^\infty$ in X such that, for every $x \in X$, there exists a unique sequence $(\lambda_n)_{n=0}^\infty$ of scalars such that $x = \sum_{n=0}^\infty \lambda_n b_n$. By X' we denote the continuous dual of X, that is the set of all continuous linear functionals on X. If X is a Banach space then we write X^* for X' with its norm defined by $\|f\| = \sup_{x\in S_X} |f(x)|$.

We write ω, c_0, c and ℓ_∞ for the sets of all complex, null, convergent and bounded sequences, $\ell_p = \{x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty\}$ for $1 \le p < \infty$, and cs and bs for the sets of

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all convergent and bounded series. By e and $e^{(n)}$ $(n = 0, 1, \dots)$, we denote the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$.

It is well known that ω is a complete locally convex linear metric space with its metric given by

$$d(x,y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x = (x_k)_{k=0}^{\infty}, y = (y_k)_{k=0}^{\infty} \in \omega; \tag{1.1}$$

c_0, c, ℓ_∞, ℓ_p $(1 \leq p < \infty)$, cs and bs are Banach spaces with their natural norms given by $\|x\|_\infty = \sup_k |x_k|$ on c_0, c and ℓ_∞, $\|x\|_p = (\sum_{k=0}^{\infty} |x_k|^p)^{1/p}$ on ℓ_p, and $\|x\|_{bs} = \sup_n |\sum_{k=0}^{n} x_k|$ on cs and bs.

Furthermore, c_0^* is norm isomorphic to ℓ_1; this means $f \in c_0^*$ if and only if $f(x) = \sum_{k=0}^{\infty} a_k x_k$ $(x \in X)$ for some $a \in \ell_1$, and $\|f\| = \|a\|$ ([Wil1, Example 6.4.4, p. 91]). Similarly ℓ_1^* is norm isomorphic to ℓ_∞ ([Wil1, Example 6.4.2, p. 91]), and ℓ_p^* for $1 < p < \infty$ is norm isomorphic to ℓ_q where $q = p/(p-1)$ ([Wil1, Example 6.4.3, p. 91]); also $f \in c^*$ if and only if

$$f(x) = \chi_f \lim_{k \to \infty} x_k + \sum_{k=0}^{\infty} a_k x_k \ (x \in c) \text{ with } a = (f(e^{(k)})_{k=0}^{\infty} \in \ell_1, \tag{1.2}$$

where

$$\chi_f = f(e) - \sum_{k=0}^{\infty} f(e^{(k)}) \text{ ([Wil1, Example 6.4.5, p. 92]);} \tag{1.3}$$

furthermore

$$\|f\| = |\chi_f| + \|a\|_1. \tag{1.4}$$

Finally, the continuous dual of ℓ_∞ is not given by a sequence space ([Wil1, Example 6.4.8, pp. 93, 94]).

We give a short survey of the most important concepts and methods of summability, an introduction to the basic theories of FK, BK, AK and AD spaces, consider multiplier and dual spaces of sequence spaces, characterise matrix transformations between sequence spaces, and apply the Hausdorff measure of noncompactness to the characterisations of compact operators between sequence spaces.

2. Summability

This section is intended as a motivation of what follows; the results presented here are not needed in the sequel.

Summability encompasses a variety of fields, originally mainly from analysis, and has many applications, for instance in numerical analysis to speed up the rate of convergence, and in approximation theory, operator theory and the theory of orthogonal series.

2.1. **Concepts and Methods of Summability.** The *classical summability theory* deals with a generalisation of the convergence of sequences or series of real or complex numbers. The idea is to assign a *limit* to divergent sequences or series by considering a *transform* rather than the original sequence or series. Most popular are *matrix transformations* given by an infinite matrix $A = (a_{nk})_{n,k=0}^{\infty}$.

There are three concepts of summability, *ordinary*, *absolute* and *strong summability.*

First we consider *ordinary summability.* A sequence $x = (x_k)_{k=0}^{\infty}$ of complex numbers is said to be *summable* A to a complex number η, if the series

$$A_n x = \sum_{k=0}^{\infty} a_{nk} x_k \text{ converge for all } n \text{ and } \lim_{n\to\infty} A_n x = \eta; \tag{2.1}$$

this is denoted by $x \to \eta(A)$. The matrix A defines a *summability method* A or a *matrix transformation* by (2.1).

Let $0 < p < \infty$. Then a sequence x is said to be *absolutely summable with index* p *to a complex number* η if the series $A_n x$ in (2.1) converge for all n, and $\sum_{n=0}^{\infty} |A_n x|^p = \eta$; this is denoted by $x \to \eta|A|^p$. A sequence x is said to be *strongly summable* A *with index* p *to a complex number* ξ if the series $\sum_{k=0}^{\infty} a_{nk}|x_k - \xi|^p$ converge for all n and $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk}|x_k - \xi|^p = 0$; this is denoted by $x \to \xi[A]^p$.

Example **2.1.** *Let the matrix* $A = (a_{n,k})_{n,k=0}^{\infty}$ *be given by* $a_{nk} = 1/(n+1)$ *for* $0 \le k \le n$ *and* $a_{nk} = 0$ $(n = 0, 1, \dots)$. *Then* A *transforms every sequence* x *into the sequence of its arithmetic means. It is well known by Cauchy's theorem that every convergent sequence is summable* A *to the same limit. Furthermore, the divergent sequence* $((-1)^k)_{k=0}^{\infty}$ *is summable* A *to* 0.

The most important summability methods are given by *Hausdorff* matrices and their special cases, the *Cesàro*, *Hölder* and *Euler* matrices, and by *Nörlund* matrices. All these matrices are *triangles*, that is $a_{nk} = 0$ for $k > n$ and $a_{nn} \neq 0$ $(n = 0, 1, \dots)$.

Let $\mu = (\mu)_{n=0}^{\infty}$ be a given complex sequence, $M = (m_{nk})_{n,k=0}^{\infty}$ be the diagonal matrix with $m_{nn} = \mu_n$ $(n = 0, 1, \dots)$, and $D = (d_{nk})_{n,k=0}^{\infty}$ be the matrix with $d_{nk} = (-1)^k \binom{n}{k}$. Then the matrix $H = H(\mu) = DMD$ is called the *Hausdorff matrix associated with the sequence* μ; its entries are given by $h_{nk} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j}\binom{j}{k} \mu_j$ $(0 \le k \le n;\ n = 0, 1, \dots)$. The *Cesàro matrix* C_α *of order* $\alpha > -1$ is the Hausdorff matrix associated with the sequence μ where $\mu_n = A_n^\alpha = \binom{n+\alpha}{n}$ $(n = 0, 1, \dots)$; its entries are given by $(C_\alpha)_{n,k} = A_{n-k}^{\alpha-1}/A_n^\alpha$ $(0 \le k \le n;\ n = 0, 1, \dots)$; the numbers A_n^α are called the *Cesàro coefficients of order* α. The *Hölder matrix* H^α *of order* $\alpha > -1$ is the Hausdorff matrix associated with the sequence μ where $\mu_n = (n+1)^{-\alpha}$ $(n = 0, 1, \dots)$; no explicit formula is known for the entries of the matrices H^α, in general. The *Euler matrix* E_q *of order* $q > 0$ is the Hausdorff matrix associated with the sequence μ where $\mu_n = (q+1)^{-n}$ $(n = 0, 1, \dots)$; its entries are given by $(E_q)_{n,k} = \binom{n}{k} q^{n-k} (q+1)^{-n}$ $(0 \le k \le n;\ n = 0, 1, \dots)$.

Finally, let $q = (q_k)_{k=0}^{\infty}$ be a sequence of complex numbers such that $Q_n = \sum_{k=0}^{n} q_k \neq 0$ for all n. Then the *Nörlund matrix* (N, q) is given by $((N, q))_{n,k} = q_{n-k}/Q_n$ $(0 \le k \le n;\ n = 0, 1, \dots)$.

Example **2.2.** *(a) Let $\mu = e$. Then we obtain for the Hausdorff matrix $H = H(\mu)$*

$$h_{nk} = \sum_{j=k}^{n} (-1)^{j+k} \binom{n}{j} \binom{j}{k} \mu_j = \binom{n}{k} \sum_{j=k}^{n} (-1)^{j+k} \binom{n-k}{j-k}$$
$$= \binom{n}{k} \sum_{j=0}^{n-k} (-1)^{k} \binom{n-k}{j} = \delta_{nk} \quad (n,k = 0,1,\dot{)}$$

where $\delta_{nn} = 1$ and $\delta_{nk} = 0$ for $k \neq 0$. Thus we have $H = I$, the identity matrix.
(b) Let $\mu_n = 1/(n+1) = A_n^1$ $(n = 0,1,\dots)$.Then we obtain $H(u) = H^1 = C_1$; thus the matrix A of Example 2.1 is a Hölder and Cesàro matrix of order 1.
(c) Let $q = e$. Then we obtain $Q_n = n+1$ $(n = 0,1,\dots)$. Thus the matrix A of Example 2.1 is also the Nörlund matrix (N,e).

We refer the interested reader to [Boo, Coo, Har, Mad, Pey, Z–B] for the classical summability theory.

2.2. **Matrix Transformations.** The theory of matrix transformations deals with establishing necessary and sufficient conditions on the entries of a matrix to map a sequence space X into a sequence space Y. This is a natural generalisation of the problem to characterise all summability methods given by infinite matrices that preserve convergence.

Given $X, Y \subset \omega$, we write (X,Y) for the class of all infinite matrices that map X into Y. So $A \in (X,Y)$ if and only if the series $A_n x$ in (2.1) converge for all n and all $x \in X$, and

$$Ax = (A_n x)_{n=0}^{\infty} \in Y \text{ for all } x \in X. \tag{2.2}$$

The first results were the *Toeplitz theorem* for the classes (c,c) of *conservative* or (*convergence preserving*) matrices, $(c,c;P)$ of *regular* matrices, that is conservative matrices that preserve limits, and the *Schur theorem* for the classes (ℓ_∞, c), the so–called *coercive* matrices, and (ℓ_∞, c_0).

Theorem 2.3 (O. Toeplitz, 1911)**.** ([Toe]) (a) We have $A \in (c,c)$ if and only if

(i) $\|A\| = \sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$, (ii) $\lim_{n\to\infty} a_{nk} = \alpha_k$ exists for every k and (iii) $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha$ exists.

If $A \in (c,c)$ and $x \in c$ then

$$\lim_{n\to\infty} A_n(x) = \left(\alpha - \sum_{k=0}^{\infty} \alpha_k\right) \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} \alpha_k x_k.$$

(b) We have $A \in (c,c;P)$ if and only if (i), (ii) and (iii) in (a) hold with $\alpha_k = 0$ $(k = 0,1,\dots)$ and $\alpha = 1$.

Theorem 2.4 (O. Schur, 1920)**.** ([Wil2, Theorem 1.7.18, p. 15]) *(a) We have $A \in (\ell_\infty, c)$ if and only if (ii) in Theorem 2.3 holds and*

(i') $\sup_n \sum_{k=0}^{\infty} |a_{nk}|$ *is uniformly convergent in n.*

(b) ([S–T, 21, (21.1)]) *We have* $A \in (\ell_\infty, c_0)$ *if and only if*

(ii')
$$\lim_{n\to\infty}\sum_{k=0}^{\infty}|a_{nk}| = 0.$$

We close this section with some applications of Theorems 2.3 and 2.4.

Example **2.5.** *(a) The matrix A of Example 2.1 is regular.*
(b) The Euler matrices E_q *are regular for all* $q > 0$.

Proof. (a) This is obvious from Theorem 2.3 (b).

(b) We write $A = E_q$. Since $a_{nk} \geq 0$ $(n, k = 0, 1, \dots)$ for $q > 0$, it follows that

$$\sum_{k=0}^{\infty}|a_{nk}| = \sum_{k=0}^{\infty}a_{nk} = \frac{1}{(q+1)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k} = \frac{(q+1)^n}{(q+1)^n} = 1 \text{ for all } n,$$

and (i) and (iii) of Theorem 2.3 (b) are satisfied. We fix k. Since $0 < q/(q+1) < 1$, we have, for $\rho = 1/q > 0$, $q/(q+1) = 1/(1+\rho)$, and so

$$\begin{aligned} 0 \leq a_{nk} &= \frac{1}{(q+1)^n}\binom{n}{k}q^{n-k} = \frac{1}{q^k}\binom{n}{k}\frac{1}{(1+\rho)^n} \leq \frac{1}{q^k}\binom{n}{k}\frac{1}{\binom{n}{k+1}\rho^{k+1}} \\ &= \frac{1}{\rho(q\rho)^k}\frac{k+1}{n-k} \to 0 \quad (n\to\infty). \end{aligned}$$

Thus (ii) of Theorem 2.3 (b) is also satisfed. □

Example **2.6.** *The famous Steinhaus theorem states that, for every regular matrix A, there is a bounded sequence which is not summable A.*

Proof. We assume there is a matrix $A \in (c, c; P) \cap (\ell_\infty, c)$. Then it follows from Theorem 2.3 (iii), (ii), and Theorem 2.4 (i') that $1 = \lim_{n\to\infty}\sum_{k=0}^{\infty}a_{nk} = \sum_{k=0}^{\infty}(\lim_{n\to\infty}a_{nk}) = 0$, a contradiction. □

Example **2.7.** *Weak and strong convergence coincide in* ℓ_1.

Proof. We assume that the sequence $(x^{(n)})_{n=0}^{\infty}$ is weakly convergent to x in ℓ_1, that is $f(x^{(n)}) - f(x) \to 0$ $(n \to \infty)$ for every $f \in \ell_1^*$. Since ℓ_1^* and ℓ_∞ are norm isomorphic, to every $f \in \ell_1^*$ there corresponds a sequence $a \in \ell_\infty$ such that $f(y) = \sum_{k=0}^{\infty}a_k y_k$ for all $y \in \ell_1$. We define the matrix $B = (b_{nk})_{n,k=0}^{\infty}$ by $b_{nk} = x_k^{(n)} - x_k$ $(n, k = 0, 1, \dots)$. Then we have $f(x^{(n)}) - f(x) = \sum_{k=0}^{\infty}a_k(x_k^{(n)} - x_k) = \sum_{k=0}^{\infty}b_{nk}a_k \to 0$ $(n \to \infty)$ for all $a \in \ell_\infty$, that is $B \in (\ell_\infty, c_0)$, and it follows from Theorem 2.4 (b) that $\|x^{(n)} - x\|_1 = \sum_{k=0}^{\infty}|x_k^{(n)} - x_k| = \sum_{k=0}^{\infty}|b_{nk}| \to 0$ $(n \to \infty)$. □

Further results on matrix transformations and references can be found in [Boo, Coo, K–G, S–T, Z–B, Wil2, Mad, M–R, Mal, J–M], and in [Mad1] for infinite matrices of operators.

3. FK, BK, AK and AD Spaces

The theory of FK spaces is the most powerful tool in the theory of matrix transformations ([Wil1, Wil2, K–G, Zel, M–R]). The fundamental result of this section is Theorem 3.8 which states that matrix maps between FK spaces are continuous.

We start with a more general definition.

Definition 3.1. *Let H be a linear space and a Hausdorff space. An FH* space *is a Fréchet space, that is a locally convex linear metric space X, such that X is a subspace of H and the topology of X is stronger than the restriction of the topology of H on X.*
If $H = \omega$ with its topology given by the metric d of (1.1), then an FH space is called an FK space.
A BH space *or a BK* space *is an FH or FK space which is a Banach space.*

Remark **3.2.** *(a) If X is an FH space, then the inclusion map $\iota : X \to H$ with $\iota(x) = x$ for all $x \in X$ is continuous. Therefore X is continuously embedded in H.*
(b) Since convergence in (ω, d) and coordinatewise convergence are equivalent ([Wil1, Theorem 4.1.1, p. 54]*), convergence in an FK space implies coordinatewise convergence.*
*(c) The letters **F**, **H**, **K** and **B** stand for **F**réchet, **H**ausdorff, **K**oordintate, the German word for coordinate, and **B**anach.*

Example **3.3.** *Let $H = \mathcal{F} = \{f : [0,1] \to I\!R\}$, and, for every $t \in [0,1]$, let $\hat{t} : \mathcal{F} \to I\!R$ be the function with $\hat{t}(f) = t(f)$. We assume that $\mathcal{F}$ has the weak topology by $\Phi = \{\hat{t} : t \in [0,1]\}$. Then $C[0,1] = \{f \in \mathcal{F} : f$ is continuous$\}$ is a BH space with $\|f\| = \sup_{t\in[0,1]} |f(t)|$.*

Proof. Let $(f_k)_{k=0}^{\infty}$ be a sequence in $C[0,1]$ with $f_k \to 0$ $(k \to \infty)$, then $\hat{t}(f_k) = f_k(t) \to 0$ $(k \to \infty)$ for all $\hat{t} \in \Phi$, that is $f_k \to 0$ $(k \to \infty)$ in $\mathcal{F}$. □

Example **3.4.** *Trivially ω is an FK space with the metric of (1.1). The spaces ℓ_∞, c and c_0 and ℓ_p $(1 \le p < \infty)$ are BK spaces with their natural norms, since $|x_k| \le \|x\|$ in each case.*

The following results are fundamental.

Theorem 3.5. ([Wil2, Theorem 4.2.2, p. 56]) *Let X be a Fréchet space, Y be an FH space and $f : X \to Y$ be linear. Then $f : X \to H$ is continuous, if and only if $f : X \to Y$ is continuous.*

Proof. Let $\mathcal{T}_X$, $\mathcal{T}_Y$ and $\mathcal{T}_H$ be the topologies on X, Y and of H on Y, respectively.
First, we assume that $f : (X, Y)$ is continuous. Since Y is an FH space, we have $\mathcal{T}_H \subset \mathcal{T}_Y$, and so $f : X \to H$ is continuous.
Conversely, we assume that $f : X \to (Y, \mathcal{T}_H)$ is continuous, then it has closed graph by the closed graph lemma (Theorem A.1). Since Y is an FH space, we again have $\mathcal{T}_H \subset \mathcal{T}_Y$, and so $f : X \to (Y, \mathcal{T}_Y)$ has closed graph. Hence $f : X \to (Y, \mathcal{T}_Y)$ is continuous by the closed graph theorem (Theorem A.2). □

We obtain as an immediate consequence of Theorem 3.5.

Corollary 3.6. ([Wil2, Corollary 4.2.3, p. 56]) *Let X be a Fréchet space, Y be an FK space, $f : X \to Y$ be linear, and $P_n : X \to \mathbb{C}$ $(n = 0, 1, \dots)$ be defined by $P_n(x) = x_n$ for all $x \in X$. If $P_n \circ f : X \to \mathbb{C}$ is continuous for every n, then $f : X \to Y$ is continuous.*

Proof. Since convergence and coordinatewise convergence are equivalent in ω by Remark 3.2 (b), the continuity of $P_n : X \to \mathbb{C}$ for all n implies the continuity of $f : X \to \omega$, hence of $f : X \to Y$ by Theorem 3.5. □

By ϕ we denote the set of all finite sequences. Thus $x = (x_k)_{k=0}^{\infty} \in \phi$ if and only if there is an integer k such that $x_j = 0$ for all $j > k$.

Theorem 3.7. ([M–R, Remark 1.16, p. 152]) *Let $X \supset \phi$ be an FK space. If the series $\sum_{k=0}^{\infty} a_k x_k$ converge for all $x \in X$, then the linear functional f_a defined by $f_a(x) = \sum_{k=0}^{\infty} a_k x_k$ for all $x \in X$ is continuous.*

Proof. We define the functionals f_a $(n \in \mathbb{N}_0)$ by $f_a^{[n]}(x) = \sum_{k=0}^{n} a_k x_k$ for all $x \in X$. Since X is an FK space and $f_a^{[n]}$ is a finite linear combination of coordinates, we have $f_a^{[n]} \in X'$ for all n. By hypothesis, the limits $f_a(x) = \lim_{n\to\infty} f_a^{[n]}(x)$ exist for all $x \in X$, hence $f_a \in X'$ by the Banach–Steinhaus theorem (Theorem A.3). □

Theorem 3.8. ([Wil2, Theorem 4.2.8, p. 57]) *Any matrix map between FK spaces is continuous.*

Proof. Let X and Y be FK spaces, $A \in (X, Y)$ and $f_A : X \to Y$ be defined by $f_A(x) = Ax$ for all $x \in X$. Since the maps $P_n \circ f_A : X \to \mathbb{C}$ are continuous for all n by Theorem 3.7, $f_A : X \to Y$ is continous by Corollary 3.6. □

It turns out that the FH topology of an FH space is unique.

Theorem 3.9. ([Wil2, Corollary 4.2.4, p. 56]) *Let X and Y be FH spaces with $X \subset Y$. Then the topology $\mathcal{T}_X$ is larger than the topology $\mathcal{T}_Y|_X$ of Y on X.*
They are equal if and only if X is a closed subspace of Y.
In particular, the topology of an FH space is unique.

Proof. Since X is an FH space, the inclusion map $\iota : X \to H$ is continuous by Remark 3.2 (a), hence $\iota : X \to Y$ is continuous by Theorem 3.5. This implies $\mathcal{T}_X \supset \mathcal{T}_Y|_X$.
Now let $\mathcal{T}$ and $\mathcal{T}'$ be FH topologies for an FK space. Then it follows by what we have just shown that $\mathcal{T} \subset \mathcal{T}' \subset \mathcal{T}$.
If X is closed in Y, then X becomes an FH space with $\mathcal{T}_Y|_X$. It follows from the uniqueness that $\mathcal{T}_X = \mathcal{T}_Y|_X$.
If $\mathcal{T}_X = \mathcal{T}_Y|_X$, then X is a complete, hence closed, subspace of Y. □

The class of FK spaces is fairly large.

Example **3.10.** A Banach sequence space which is not a BK space
We consider the spaces $(c_0, \|\cdot\|_\infty)$ and $(\ell_2, \|\cdot\|_2)$. Since they have the same algebraic dimension, there is an isomorphism $f : c_0 \to \ell_2$. We define a second norm $\|\cdot\|$ on c_0 by $\|x\| = \|f(x)\|_2$ for all $x \in c_0$. Then $(c_0, \|\cdot\|)$ becomes a Banach space. But c_0 and ℓ_2 are not linearly homeomorphic, since ℓ_2 is reflexive, and c_0 is not. Therefore the two

norms on c_0 are incomparable. By Example 3.4 and Theorem 3.9, $(c_0, \|\cdot\|)$ is a Banach sequence space which is not a BK space.

Theorem 3.11. ([Wil2, Theorem 4.2.5, p. 57]) *Let X, Y and Z be FH spaces with $X \subset Y \subset Z$. If X is closed in Z, then X is closed in Y.*

Proof. Since X is closed in $(Y, \mathcal{T}_Z|_Y)$, it is closed in $(Y, \mathcal{T}_Y)$ by Theorem 3.9. □

Let Y be a topological space, and $E \subset Y$. Then we write $\mathrm{cl}_Y(E)$ for the closure of E in Y.

Theorem 3.12. ([Wil2, Theorem 4.2.7, p. 57]) *Let X and Y be FH spaces with $X \subset Y$, and E be a subset of X. Then we have*

$$cl_Y(E) = cl_Y(cl_X(E)), \text{ in particular } cl_X(E) \subset cl_Y(E).$$

Proof. Since $\mathcal{T}_Y|_X \subset \mathcal{T}_X$ by Theorem 3.9, it follows that $\mathrm{cl}_X(E) \subset \mathrm{cl}_Y(E)$. This implies

$$\mathrm{cl}_Y(\mathrm{cl}_X(E)) \subset \mathrm{cl}_Y(\mathrm{cl}_Y(E)) = \mathrm{cl}_Y(E).$$

Conversely, $E \subset \mathrm{cl}_X(E)$ implies $\mathrm{cl}_Y(E) \subset \mathrm{cl}_Y(\mathrm{cl}_X(E))$. □

Example **3.13.** *(a) Since c_0 and c are closed in ℓ_∞, their BK topologies are the same; since ℓ_1 is not closed in ℓ_∞, its BK topology is strictly stronger than that of ℓ_∞ on ℓ_1 (Theorem 3.9).*
(b) If c is not closed in an FK space X, then X must contain unbounded sequences (Theorem 3.11).

Definition 3.14. *Let $X \supset \phi$ be an FK space. Then X is said to have*
(a) AD if $cl_X(\phi) = X$;
(b) AK if every sequence $x = (x_k)_{k=0}^\infty \in X$ has a unique representation $x = \sum_{k=0}^\infty x_k e^{(k)}$.

Remark **3.15.** *The letters **A**, **D** and **K** stand for **a**bschnitts**d**icht, the German word for sectionally dense, and **A**bschnitts**k**onvergenz, the German word for sectional convergence.*

Example **3.16.** *(a) Every FK space with AK obviously has AD.*
(b) An Example of an FK space with AD which does not have AK can be found in [Wil2, Example 5.2.14, p. 80].
(c) The spaces ω, c_0 and ℓ_p $(1 \le p < \infty)$ have AK.
(d) The space c does not have AK; every sequence $x = (x_k)_{k=0}^\infty \in c$ has a unique representation $x = \xi\, e + \sum_{k=0}^\infty (x_k - \xi)e^{(k)}$ where $\xi = \lim_{k\to\infty} x_k$.
(e) The space ℓ_∞ has no Schauder basis, since it is not separable.

Theorem 3.17. ([Wil2, 8.3.6, p. 123]) *Let X be an FK space with AD, and Y and Y_1 be FK spaces with Y_1 a closed subspace of Y. Then $A \in (X, Y_1)$ if and only if $A \in (X, Y)$ and $Ae^{(k)} \in Y_1$ for all k.*

Proof. First, we assume $A \in (X, Y_1)$. Then $Y_1 \subset Y$ implies $A \in (X, Y)$, and $e^{(k)} \in X$ for all k implies $Ae^{(k)} \in Y_1$ for all k.
Conversely, we assume $A \in (X, Y)$ and $Ae^{(k)} \in Y_1$ for all k. We define the map $f_A : X \to Y$ by $f_A(x) = Ax$ for all $x \in X$. Then $Ae^{(k)} \in Y_1$ implies $f_A(\phi) \subset Y_1$. By Theorem 3.8,

f_A is continuous, hence $f_A(\text{cl}_X(\phi)) = \text{cl}_Y(f_A(\phi))$. Since Y_1 is closed in Y, and ϕ is dense in the AD space X, we have $f_A(X) = f_A(\text{cl}_X(\phi)) = \text{cl}_Y(f_A(\phi)) \subset \text{cl}_Y(Y_1) = \text{cl}_{Y_1}(Y_1) = Y_1$ by Theorem 3.9. □

Theorem 3.18. ([Wil2, 8.3.7, p. 123]) *Let X be an FK space, $X_1 = X \oplus e = \{x_1 = x + \lambda e : x \in X,\ \lambda \in \mathbb{C}\}$, and Y be a linear subspace of ω. Then $A \in (X_1, Y)$ if and only if $A \in (X, Y)$ and $Ae \in Y$.*

Proof. First, we assume $A \in (X_1, Y)$. Then $X \subset X_1$ implies $A \in (X, Y)$, and $e \in X_1$ implies $Ae \in Y$.
Conversely, we assume $A \in (X, Y)$ and $Ae \in Y$. Let $x_1 \in X_1$ be given. Then there are $x \in X$ and $\lambda \in \mathbb{C}$ such that $x_1 = x + \lambda e$, and it follows that $Ax_1 = A(x + \lambda e) = Ax + \lambda Ae \in Y$. □

We close this section with two applications of our results.

Let (X, d) be a metric space, $\delta > 0$ and $x_0 \in X$. Then we write $B_\delta[x_0] = \{x \in X : d(x, x_0) \leq \delta\}$ for the closed ball of radius δ with its centre in x_0. If $X \subset \omega$ is a linear metric space and $a \in \omega$, then we write

$$\|a\|_\delta^* = \|a\|_{X,\delta}^* = \sup_{x \in B_\delta[0]} \left| \sum_{k=0}^{\infty} a_k x_k \right|,$$

provided the expression on the right hand exists and is finite which is the case whenever the series $\sum_{k=0}^{\infty} a_k x_k$ converge for all $x \in X$ (Theorem 3.7); if X is a normed space then we write

$$\|a\|^* = \|a\|_{X^*} = \sup_{x \in S_X} \left| \sum_{k=0}^{\infty} a_k x_k \right|.$$

The first result is the characterisation of the class (X, ℓ_∞) for arbitrary FK spaces X.

Theorem 3.19. ([M–R, Theorem 1.23 (b)]) *Let X be an FK space. Then we have $A \in (X, \ell_\infty)$ if and only if*

$$\|A\|_\delta^* = \sup_n \|A_n\|_\delta^* < \infty \text{ for some } \delta > 0, \tag{3.1}$$

where $A_n = (a_{nk})_{k=0}^{\infty}$ denotes the sequence in the n–th row of the matrix A.

Proof. First, we assume that (3.1) is satisfied. Then the series $A_n x$ converge for all $x \in B_\delta[0]$ and for all n, and $Ax \in \ell_\infty$ for all $x \in B_\delta[0]$. Since the set $B_\delta[0]$ is absorbing by [Wil1, Fact (ix), p. 53], we conclude that the series $A_n x$ converge for all n and all $x \in X$, and $Ax \in \ell_\infty$ for all x.
Conversely, we assume $A \in (X, \ell_\infty)$. Then the map $L_A : X \to \ell_\infty$ defined by

$$L_A(x) = Ax \text{ for all } x \in X \tag{3.2}$$

is continuous by Theorem 3.8. Hence there exist a neighbourhood N of 0 in X and a real $\delta > 0$ such that $B_\delta[0] \subset N$ and $\|L_A(x)\|_\infty < 1$ for all $x \in X$. This implies (3.1). □

Theorem 3.20. ([M–R, Theorem 1.23. p. 155]) *Let X and Y be BK spaces.*
(a) Then $(X,Y) \subset \mathcal{B}(X,Y)$, that is every $A \in (X,Y)$ defines an operator $L_A \in \mathcal{B}(X,Y)$ by (3.2).
(b) If X has AK then $\mathcal{B}(X,Y) \subset (X,Y)$.
(c) We have $A \in (X,\ell_\infty)$ if and only if

$$\|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty; \tag{3.3}$$

if $A \in (X,\ell_\infty)$ then

$$\|L_A\| = \|A\|_{(X,\ell_\infty)}. \tag{3.4}$$

Proof. (a) This is Theorem 3.8.

(b) Let $L \in \mathcal{B}(X,Y)$ be given. We write $L_n = P_n \circ L$ for all n, and put $a_{nk} = L_n(e^{(k)})$ for all n and k. Let $x = (x_k)_{k=0}^\infty \in X$ be given. Since X has AK, we have $x = \sum_{k=0}^\infty x_k e^{(k)}$, and since Y is a BK space, it follows that $L_n \in X^*$ for all n. Hence we obtain $L_n(x) = \sum_{k=0}^\infty x_k L_n(e^{(k)}) = \sum_{k=0}^\infty a_{nk}x_k = (Ax)_n$ for all n, and so $L(x) = Ax$.

(c) This follows immediately from Theorem 3.19 and the definition of $\|A\|_{(X,\ell_\infty)}$. □

4. Multiplier and Dual Spaces

The so–called β–duals are of greater interest than the continuous duals in the theory of matrix transformations. They naturally arise in the characterisations of matrix transformations in connection with the convergence of the series $A_n x$.

The β–duals of sequence spaces are special cases of multiplier spaces.

Definition 4.1. *Let X and Y be subsets of ω. The set $M(X,Y) = \{a \in \omega : ax = (a_k x_k)_{k=0}^\infty \in Y$ for all $x \in X\}$ is called the* multiplier space of X in Y. *Special cases are $X^\alpha = M(X,\ell_1)$, $X^\beta = M(X,cs)$ and $X^\gamma = M(X,bs)$, the α–, β and γ–* duals of X.

Proposition 4.2. ([M–R, Lemma 1.25, p. 156]) *Let X, X_1, Y, $Y_1 \subset \omega$ and $\{X_\delta\}$ be a collection of subsets of ω. Then we have*

(i) $Y \subset Y_1$ *implies* $M(X,Y) \subset M(X,Y_1)$
(ii) $X \subset X_1$ *implies* $M(X_1,Y) \subset M(X,Y)$
(iii) $X \subset M(M(X,Y),\, Y)$
(iv) $M(X,Y) = M\left(M(M(X,Y),\, Y),\, Y\right)$
(v) $M\left(\bigcup_\delta X_\delta, Y\right) = \bigcap_\delta M(X_\delta, Y)$.

Proof. (i), (ii) Parts (i) and (ii) are trivial.

(iii) If $x \in X$, then $ax \in Y$ for all $a \in M(X,Y)$, and consequently $x \in M(M(X,Y),Y)$.

(iv) We replace X by $M(X,Y)$ in (iii) to obtain $M(X,Y) \subset M(M(M(X,Y),Y),Y)$. Conversely we have $X \subset M(M(X,Y))$ by (iii), so $M(M(M(X,Y),Y),Y) \subset M(X,Y)$ by (ii).

(v) First $X_\delta \subset \bigcup_\delta X_\delta$ for all δ implies $M(\bigcup_\delta X_\delta,\, Y) \subset \bigcap_\delta M(X_\delta,Y)$ by (ii). Conversely, if $a \in \bigcap_\delta M(X_\delta,Y)$, then $a \in M(X_\delta,Y)$ for all δ, and so we have $ax \in Y$ for all $x \in X_\delta$ and all δ. This implies $ax \in Y$ for all $x \in \bigcup_\delta X_\delta$, hence $a \in M(\bigcup_\delta X_\delta,\, Y)$. □

Example **4.3.** *We have (i) $M(c_0, c) = \ell_\infty$; (ii) $M(c, c) = c$; (iii) $M(\ell_\infty, c) = c_0$.*

Proof. (i) If $a \in \ell_\infty$, then $ax \in C$ for all $x \in c_0$, and so $\ell_\infty \subset M(c_0, c)$.
Conversely, we assume $a \notin \ell_\infty$. Then there is a subsequence $a_{k(j)}$ of the sequence a such that $|a_{k(j)}| > j + 1$ for all $j = 0, 1, \dots$. We define the sequence x by

$$x_k = \begin{cases} \dfrac{(-1)^j}{a_{k(j)}} & (k = k(j)) \\ 0 & (k \neq k(j)) \end{cases} \quad (j = 0, 1, \dots). \tag{4.1}$$

Then we have $x \in c_0$ and $a_{k(j)}x_{k(j)} = (-1)^j$ for all $j = 0, 1, \dots$, hence $ax \notin c$. This shows $M(c_0, c) \subset \ell_\infty$.

(ii) If $a \in c$, then $ax \in c$ for all $x \in c$, and so $c \subset M(c, c)$.
Conversely, we assume $a \notin c$. Since $e \in c$ and $ae = a \notin c$, we have $a \notin M(c, c)$. This shows $M(c, c) \subset c$.

(iii) If $a \in c_0$ then $ax \in c$ for all $x \in \ell_\infty$, and so $c_0 \subset M(\ell_\infty, c_0)$.
Conversely, we assume $a \notin c_0$. Then there are a real $b > 0$ and a subsequence $(a_{k(j)})_{j=0}^\infty$ of the sequence a such that $|a_{k(j)}| > b$ for all $j = 0, 1, \dots$. We define the sequence x as in (4.1). Then we have $x \in \ell_\infty$ and $a_{k(j)}x_{k(j)} = (-1)^j$ for all $j = 0, 1, \dots$, hence $a \notin M(\ell_\infty, c)$. This shows $M(\ell_\infty, c) = c_0$. □

Example **4.4.** *Let † denote any of the symbols α, β or γ. Then we have $\omega^\dagger = \phi$, $\phi^\dagger = \omega$, $c_0^\dagger = c^\dagger = \ell_\infty^\dagger = \ell_1$, $\ell_1^\dagger = \ell_\infty$, and $\ell_p^\dagger = \ell_q$ $(1 < p < \infty; q = p/(p-1))$.*

Another dual space frequently arises in the theory of sequence spaces.

Definition 4.5. *Let $X \supset \phi$ be an FK space. Then $X^f = \{(f(e^{(n)}))_{n=0}^\infty : f \in X'\}$ is called the* functional dual of X.

Theorem 4.6. *(a) We have $X^\alpha \subset X^\beta \subset X^\gamma$ and $X \subset X^{\dagger\dagger}$ where † is any of the symbols α, β and γ.*
(b) Let $X \supset \phi$ be an FK space. Then we have $X^f = (cl_X(\phi))^f$ ([Wil2, Theorem 7.2.4, p. 106]).
(c) Let $X, Y \supset \phi$ be FK spaces. If $X \subset Y$ then $X^f \supset Y^f$. If X is closed in Y then $X^f = Y^f$ ([Wil2, Theorem 7.2.4, p. 106]).

Proof. (a) Since $\ell_1 \subset cs \subset bs$, it follows from Proposition 4.2 (i) that $X^\alpha \subset X^\beta \subset X^\gamma$, and Proposition 4.2 (iii) yields $X \subset X^{\dagger\dagger}$.

(b) We write $Z = \text{cl}_X(\phi)$.
First, we assume that $a \in X^f$, that is $a_n = f(e^{(n)})$ $(n = 0, 1, \dots)$ for some $f \in X'$. We write $g = f|_Z$ for the restriction of f to Z. Then $a_n = g(e^{(n)})$ for all $n = 0, 1, \dots$, $g \in Z'$ and so $a \in Z^f$.
Conversely, let $a \in Z$, then $a_n = g(e^{(n)})$ $(n = 0, 1, \dots)$ for some $g \in Z'$. By the Hahn–Banach–Theorem (Theorem A.4), g can be extended to $f \in X'$, and we have $a_n = f(e^{(n)})$ for $n = 0, 1, \dots$, hence $a \in X^f$.

(c) We assume that $a \in Y^f$. Then $a_n = f(e^{(n)})$ $(n = 0, 1, \dots)$ for some $f \in Y'$. Since $X \subset Y$, we have $g = f|_X \in X'$ by Theorem 3.9. If X is closed in Y, then the FK

topologies are the same by Theorem 3.9, and we obtain $X^f = (\mathrm{cl}_X(\phi))^f = (\mathrm{cl}_Y(\phi))^f = Y^f$ from Part (b). □

It might be expected from $X \subset X^{\dagger\dagger}$ that X is contained in X^{ff}; but this is not the case in general (Example 4.7). We will, however, see below that $X \subset X^{ff}$ for BK spaces with AD (Theorem 4.16).

Example **4.7.** *Let $X = c_0 \oplus z$ with z unbounded. Then X is a BK space, $X^f = \ell_1$ and $X^{ff} = \ell_\infty$, so $X \not\subset X^{ff}$.*

Theorem 4.8. ([Wil2, Theorem 7.2.7, p. 106]) *Let $X \supset \phi$ be an FK space.*
(a) We have $X^\gamma \subset X^f$.
(b) If X has AK, then $X^\beta = X^f$.
(c) If X has AD then $X^\beta = X^\gamma$.

Proof. Let $a \in X^\beta$. We define the linear functional f by $f(x) = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$. Then $f \in X'$ by Theorem 3.7, and we have $f(e^{(n)}) = a_n$ for all n, hence $a \in X^f$. Thus we have shown

$$X^\beta \subset X^f. \tag{4.2}$$

(b) Now we assume that X has AK, and $a \in X^f$. Let $x \in X$ be given. Then $x = \sum_{k=0}^\infty x_k e^{(k)}$, since X has AK, and since $f \in X'$, we have $f(x) = f(\sum_{k=0}^\infty x_k e^{(k)}) = \sum_{k=0}^\infty x_k f(e^{(k)}) = \sum_{k=0}^\infty x_k a_k$, hence $a \in X^\beta$. Thus we have shown $X^f \subset X^\beta$. Together with $X^\beta \subset X^f$, this yields $X^\beta = X^f$.

(c) Now we assume that X has AD and $a \in X^\gamma$. We define the linear functionals f_n for $n = 0, 1, \dots$ by $f_n(x) = \sum_{k=0}^n a_k x_k$ $(x \in X)$. Since X is an FK space, we have $f_n \in X'$ for all n. Furthermore, $a \in X^\gamma$ implies that the sequence $(f_n)_{n=0}^\infty$ is pointwise bounded, hence equicontinuous by the uniform boundedness principle (Theorem A.5). Since $\lim_{n\to\infty} f_n(x)$ exists for all $x \in X$ and X has AD, it must exists for all $x \in X$ by the convergence lemma (Theorem A.6), hence $a \in X^\beta$. Thus we have shown $X^\gamma \subset X^\beta$. We also have $X^\beta \subset X^\gamma$ by Theorem 4.6 (a), hence $X^\beta = X^\gamma$.

(a) First we observe that $\mathrm{cl}_X(\phi) \subset X$ implies $X^\gamma \subset (\mathrm{cl}_X(\phi))^\gamma$ by Proposition 4.2 (ii). Furthermore, we have $(\mathrm{cl}_X(\phi))^\gamma = (\mathrm{cl}_X(\phi))^\beta \subset (\mathrm{cl}_X(\phi))^f = X^f$ by Part (c), (4.2) and Theorem 4.6 (b). Thus we have shown $X^\gamma \subset X^f$. □

Now we establish a relationship between the $\beta-$ and continuous duals of an FK space.

Theorem 4.9. ([Wil2, Theorem 7.2.9, p. 107]) *Let $X \supset \phi$ be an FK space. Then $X^\beta \subset X'$; this means, that there is a linear one–to–one map $T : X^\beta \to X'$. If X has AK then T is onto.*

Proof. We define the map T by $Ta = f_a$ $(a \in X^\beta)$ where f_a is the functional with $f_a = \sum_{k=0}^\infty a_k x_k$ for all $x \in X$, and observe that $Ta = f_a \in X'$ for all $a \in X^\beta$ by Theorem 3.7. Obviously T is linear. Furthermore, if $Ta = 0$ then $f_a(x) = \sum_{k=0}^\infty a_k x_k = 0$ for all $x \in X$, in particular $f_a(e^{(k)}) = a_k = 0$ for all k, that is $a = 0$. Thus $Ta = 0$ implies $a = 0$, and consequently T is one–to–one.
Now we assume that X has AK. Let $f \in X'$ be given. We define the sequence a by

$a_k = f(e^{(k)})$ for $k = 0, 1, \dots$. Let $x \in X$ be given. Then $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, since X has AK, and $f \in X'$ implies $f(x) = f(\sum_{k=0}^{\infty} x_k e^{(k)}) = \sum_{k=0}^{\infty} x_k f(e^{(k)}) = \sum_{k=0}^{\infty} x_k a_k$, hence $a \in X^\beta$ and $Ta = f$. This shows that the map T is onto. □

A relation between the functional and continuous duals of an FK space is given by

Theorem 4.10. *Let $X \supset \phi$ be an FK space.*
(a) Then the map $q : X' \to X^f$ given by $q(f) = (f(e^{(k)})_{k=0}^{\infty}$ is onto. Moreover, if $T : X^\beta \to X'$ denotes the map of Theorem 4.9, then $q(Ta) = a$ for all $a \in X^\beta$ ([Wil2, Theorem 7.2.10, p. 107]).
(b) Then $X^f = X'$, that is the map q of Part (a) is one–to–one, if and only if X has AD ([Wil2, Theorem 1.11.12, p. 108]).

Proof. (a) Let $a \in X^f$ be given. Then there is $f \in X'$ such that $a_k = f(e^{(k)})$ for all k, and so $q(f) = (f(e^{(k)})_{k=0}^{\infty} = a$. This shows that q is onto.
Now let $a \in X^\beta$ be given. We put $f = Ta \in X'$ and obtain $q(Ta) = q(f) = (f(e^{(k)}))_{k=0}^{\infty} = ((Ta)(e^{(k)}))_{k=0}^{\infty} = (a_k)_{k=0}^{\infty} = a$.

(b) First we assume that X has AD. Then $q(f) = 0$ implies $f = 0$ on ϕ, hence $f = 0$, since X has AD. This shows that q is one–to–one.
Conversely we assume that X does not have AD. By the Hahn–Banach theorem, (Theorem A.4) there exists an $f \in X'$ with $f \neq 0$ and $f = 0$ on ϕ. Then we have $q(f) = 0$, and q is not one–to–one. □

Example **4.11.** *We have $c^\beta = c^f = \ell_1$. The map T of Theorem 4.9 is not onto. We consider* $\lim \in X'$. *If there were $a \in X^f$ with $\lim a = \sum_{k=0}^{\infty} a_k x_k$ then it would follow that $a_k = \lim e^{(k)} = 0$, hence $\lim x = 0$ for all $x \in c$, contradicting $\lim e = 1$. Also then map q of Theorem 4.10 is not onto, since $q(\lim) = 0$.*

It turns out that the multiplier spaces and the functional duals of BK spaces are again BK spaces. These results do not extend to FK spaces, in general.

Theorem 4.12. ([M–R, Theorem 1.30, p. 158]) *Let $X \supset \phi$ and Y be BK spaces. Then $Z = M(X, Y)$ is a BK space with $\|z\| = \sup_{x \in S_X} \|xz\|$ for $z \in Z$.*

Proof. Let $\|\cdot\|_X$ and $\|\cdot\|_Y$ denote the BK norms of X and Y.
Every $z \in Z$ defines a diagonal matrix map $\hat{z} : X \to Y$ where $\hat{z}(x) = xz = (x_k z_k)_{k=0}^{\infty}$ for all $x \in X$, and $\hat{z} \in B(X, Y)$ by Theorem 3.20 (a). This embeds Z in $B(X, Y)$, for if $\hat{z} = 0$ then $(\hat{z}(e^n))_n = z_n = 0$ for all n, hence $z = 0$.
To see that the coordinates are continuous, we fix n and put $u = 1/\|e^{(n)}\|_X$ and $v = \|e^{(n)}\|_Y$. Then we have $\|ue^{(n)}\|_X = 1$ and $uv|z_n| = u\|z_n e^{(n)}\|_Y = u\|e^{(n)}z\|_Y = \|(ue^{(n)})z\|_Y \leq \|z\|$.
It remains to show that Z is a closed subspace of the Banach space $B(X, Y)$. Let $(\hat{z}^{(m)})_{m=0}^{\infty}$ be a sequence in $B(X, Y)$ with $\hat{z}^{(m)} \to T \in B(X, Y)$ $(m \to \infty)$. For every fixed $x \in X$, we obtain $\hat{z}^{(m)}(x) \to T(x) \in Y$ $(m \to \infty)$, and since Y is a BK space, this implies $x_k z_k^{(m)} = (\hat{z}^{(m)}(x))_k \to (T(x))_k$ $(m \to \infty)$ for every fixed k. If we choose $x = e^{(k)}$ then we obtain $z_k^{(m)} \to t_k = (T(e^{(k)}))_k$. Thus we have $x_k z_k^{(m)} \to x_k t_k$ and

$x_k z_k^{(m)} \to (T(x))_k$ $(m \to \infty)$, hence $T(x) = xt$, and so $T = \hat{t}$. This shows that Z is closed. □

We obtain as an immediate consequence of Theorem 4.12.

Corollary 4.13. ([M–R, Corollary 1.31, p. 158]) *The $\alpha-$, $\beta-$ and $\gamma-$duals of a BK space X are BK spaces with $\|a\|_\alpha = \sup_{x \in S_X} \|ax\|_1 = \sup_{x \in S_X}(\sum_{k=0}^\infty |a_k x_k|)$ for all $a \in X^\alpha$, and $\|a\|_\beta = \sup_{x \in S_X} \|a\|_{bs} = \sup_{x \in S_X}(\sup_n |\sum_{k=0}^n a_k x_k|)$ for all $a \in X^\beta, X^\gamma$. Furthermore, X^β is a closed subspace of X^γ.*

Proof. The first part is an immediate consequence of Theorem 4.12.
Since the BK norms on X^β and X^γ are the same and $X^\beta \subset X^\gamma$ by Theorem 4.6, the second part follows from Theorem 3.9. □

Theorem 4.12 fails to hold for FK spaces, in general.

Example **4.14.** *The space ω is an FK space, and $\omega^\alpha = \omega^\beta = \omega^\gamma = \phi$, but ϕ has no Fréchet metric.*

We give the following result without proof.

Theorem 4.15. ([Wil2, Theorem 7.2.14, p. 108]) *Let $X \supset \phi$ be a BK space. Then X^f is a BK space.*

Theorem 4.16. ([Wil2, Theorem 7.2.15, p. 108]) *Let $X \supset \phi$ be a BK space. Then $X^{ff} \supset cl_X(\phi)$. Hence, if X has AD, then $X \subset X^{ff}$.*

Proof. First we have to show $\phi \subset X^f$ in order for X^{ff} to be meaningful.
This is true because $P_k \in X'$ for all k where $P_k(x) = x_k$ $(x \in X)$ since X is a BK space, and $q(P_k) = e^{(k)}$ (Theorem 4.10 (a)).
Since the second part is equivalent to the first part by Theorem 4.6 (b), we assume that X has AD, and have to show $X \subset X^{ff}$.
Let $x \in X$ be given. We define the functional $f : X' \to \mathbb{C}$ by $f(\psi) = \psi(x)$ for all $\psi \in X'$. Then we have $|f(\psi)| = |\psi(x)| \leq \|\psi\|\|x\|$, and consequently $f \in X''$. Let $q : X' \to X^f$ be the map of Theorem 4.10 (a) which is an isomorphism by Theorem 4.10 (b), since X has AK. Thus the inverse map $q^{-1} : X^f \to X'$ exists. We define the map $g : X^f \to \mathbb{C}$ by $g(b) = \psi(x)$ $(b \in X^f)$ where $x = q^{-1}(b)$. It follows that

$$|g(b)| = |\psi(x)| = |f(\psi)| \leq \|f\|\,\|\psi\| = \|f\|\,\|q^{-1}(b)\| \leq \|f\|\,\|q^{-1}\|\,\|b\|,$$

and the open mapping theorem (Theorem A.7) yields $\|q^{-1}\| < \infty$. Thus we have $g \in (X^f)'$. Finally it follows that $x_k = P_k(x) = g(q(P_k)) = g(e^{(k)})$ for all k, hence $x \in X^{ff}$. Thus we have shown $X \subset X^{ff}$. □

The condition that X has AD is not necessary for $X \subset X^{ff}$, in general.

Example **4.17.** *Let $X = c_0 \oplus z$ with $z \in \ell_\infty$. Then we have $X^{ff} = \ell_1^f = \ell_\infty \supset X$, but X does not have AD.*

5. Matrix Transformations

We apply the results of the previous sections to give necessary and sufficient conditions on the entries of a matrix A to be in a class (X, Y).

The first two results concern the transpose A^T of a matrix A.

Theorem 5.1. ([Wil2, Theorem 8.3.8, p. 124]) *Let X be an FK space and Y be any set of sequences. If $A \in (X, Y)$ then $A^T \in (Y^\beta, X^f)$. If X and Y are BK spaces and Y^β has AD then we have $A^T \in (Y^\beta, cl_{X^f}(X^\beta))$.*

Proof. Let $A \in (X, Y)$ and $z \in Y^\beta$ be given. We define the functional $f : X \to \mathbb{C}$ by $f(x) = \sum_{n=0}^\infty z_n A_n x$ $(x \in X)$. Since X is an FK space, $Ax \in Y$ by assumption and $z \in Y^\beta$, we have $f \in X'$ by Theorem 3.7. Furthermore it follows that $f(e^{(k)}) = \sum_{n=0}^\infty z_n a_{nk} = A_k^T z$ for all k, hence $A^T z \in X^f$. This shows that $A^T \in (Y^\beta, X^f)$.
Now we assume that X and Y are BK spaces and Y has AD. Then $X^\beta \subset X^f$ by Theorems 4.6 (a) and 4.8 (a), and X^f is a BK space by Theorem 4.15. Also $\mathrm{cl}_{X^f}(X^\beta)$ is a closed subspace of X^f. Since $A \in (X, Y)$, we have $A_n = (a_{nk})_{k=0}^\infty \in X^\beta$ for all n, but $A^T e^{(n)} = (\sum_{j=0}^\infty a_{jk} e_j^{(n)})_{k=0}^\infty = (a_{nk})_{k=0}^\infty = A_n$ for all n. So we have $A^T e^{(n)} \in X^\beta$ for all n, and this and $A \in (Y^\beta, X^f)$ imply $A^T \in (Y^\beta, \mathrm{cl}_{X^f}(X^\beta))$ by Theorem 3.17. □

Theorem 5.2. ([Wil2, Theoren 8.3.9, p. 124]) *Let X and Z be BK spaces with AK and $Y = Z^\beta$. Then we have $(X, Y) = (X^{\beta\beta}, Y)$; furthermore $A \in (X, Y)$ if and only if $A^T \in (Z, X^\beta)$.*

Proof. Since X is a BK space with AK, X^β is a BK space by Corollary 4.13, and $X^\beta = X^f$ by Theorem 4.8 (b).
First we assume $A \in (X, Y)$. Then it follows by Theorem 5.1 and since $Z^{\beta\beta} \supset Z$ by Theorem 4.6 (a) that $A^T \in (Y^\beta, X^\beta) = (Z^{\beta\beta}, X^\beta) \subset (Z, X^\beta)$.
Conversely, if $A^T \in (Z, X^\beta)$ then it follows by Theorem 5.1 and since $X^{\beta\beta} \supset X$ by Theorem 4.6 (a) that $A \in (X^{\beta\beta}, Z^\beta) \subset (X, Z^\beta) = (X, Y)$. This proves the second part.
To prove the first part, we first observe that $X \subset X^{\beta\beta}$ implies $(X^{\beta\beta}, Y) \subset (X, Y)$.
Conversely we assume $A \in (X, Y)$. Then we have $A^T \in (Z, X^\beta)$ as proved above, and Theorem 5.1 implies $A = A^{TT} \in (X^{\beta\beta}, Z^\beta) = (X^{\beta\beta}, Y)$. □

Remark **5.3.** *The results of the previous sections yield the characterisations of the classes (X, Y) where X and Y are any of the spaces ℓ_p $(1 \le p \le \infty)$, c_0, c with the exceptions of (ℓ_p, ℓ_r) where both $p, r \neq 1, \infty$ (the characterisations are unknown), and of (ℓ_∞, c)* (Schur's theorem 2.4) *and (ℓ_∞, c_0)* ([S–T, **21 (21.1)**]) *for which no functional analytic proofs seem to be known.*

The class (ℓ_2, ℓ_2) was characterised by Crone ([Cro] or [Ruc, pp. 111–115]).

Example **5.4.** *(a) We have $(c_0, \ell_\infty) = (c, \ell_\infty) = (\ell_\infty, \ell_\infty)$; furthermore $A \in (\ell_\infty, \ell_\infty)$ if and only if*

$$\|A\|_{(\infty,\infty)} = \sup_n \sum_{k=0}^\infty |a_{nk}| < \infty. \tag{5.1}$$

If A is in any of the classes above then $\|L_A\| = \|A\|_{(\infty,\infty)}$.
(b) We have $A \in (c_0, c)$ if and only if (5.1) holds and

$$\lim_{n\to\infty} a_{nk} = \alpha_k \text{ exists for every } k. \tag{5.2}$$

If $A \in (c_0, c)$ then

$$\lim_{n\to\infty} A_n(x) = \sum_{k=0}^{\infty} \alpha_k x_k. \tag{5.3}$$

(c) (Toeplitz's theorem 2.3) *We have $A \in (c, c)$ if and only if (5.1) and (5.2) hold, and*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = \alpha \text{ exists.} \tag{5.4}$$

If $A \in (c, c)$ and $x \in c$ then

$$\lim_{n\to\infty} A_n(x) = \left(\alpha - \sum_{k=0}^{\infty} \alpha_k\right) \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} \alpha_k x_k. \tag{5.5}$$

Furthermore have $A \in (c, c; P)$ if and only if (5.1), (5.2) and (5.4) hold with $\alpha_k = 0$ $(k = 0, 1, \dots)$ and $\alpha = 1$.

Proof. (a) We have $A \in (c_0, \ell_\infty)$ if and only if (5.1) by (3.3) in Theorem 3.20, and since $c_0^\beta = \ell_1$ and c_0^* and ℓ_1 are norm isomorphic.
Furthermore $c_0 \subset c \subset \ell_\infty$ implies $(\ell_\infty, \ell_\infty) \subset (c, \ell_\infty) \subset (c_0, \ell_\infty)$.
Also $(c_0, \ell_\infty) = (c_0^{\beta\beta}, \ell_\infty) = (\ell_\infty, \ell_\infty)$ by the first part of Theorem 5.2.
The last part is obvious from Theorem 3.20.

(b) Since c is a closed subspace of ℓ_∞, the characterisation of the class (c_0, c) is an immediate consequence of Theorem 3.17 and Part (a).
Now we assume $A \in (c_0, c)$, and write $\|A\| = \|A\|_{(\ell_\infty,\ell_\infty)}$, for short. Let m be a given non–negative integer Then it follows from (5.2) and (5.1) that $\sum_{k=0}^{m} |\alpha_k| = \lim_{n\to\infty} \sum_{k=0}^{m} |a_{nk}|$ $\le \|A\|$. Since m was arbitrary, we have $(\alpha_k)_{k=0}^\infty \in \ell_1$,

$$\sum_{k=0}^{\infty} |\alpha_k| \le \|A\| \text{ and } \sum_{k=0}^{\infty} |\alpha_k x_k| < \|A\|\,\|x\|_\infty \text{ for all } x \in c. \tag{5.6}$$

Now let $x \in c_0$ and $\varepsilon > 0$ be given. Then we can choose an integer $k(\varepsilon)$ such that $|x_k| \le \varepsilon/((4\|A\| + 1))$ for all $k > k(\varepsilon)$, and by (5.2) we can choose and integer $n(\varepsilon)$ such that $\sum_{k=0}^{k(\varepsilon)} |a_{nk} - \alpha_k|\,|x_k| < \varepsilon/2$ for all $n > n(\varepsilon)$. Let $n > n(\varepsilon)$. Then (5.1) and (5.6) imply

$$\begin{aligned}\left|A_n x - \sum_{k=0}^{\infty} \alpha_k x_k\right| &\le \sum_{k=0}^{k(\varepsilon)} |a_{nk} - \alpha_k| + \sum_{k=k(\varepsilon)}^{\infty} (|a_{nk} + \alpha_k|)|x_k| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4\|A\| + 1}\left(\sum_{k=0}^{\infty} |a_{nk}| + \sum_{k=0}^{\infty} |\alpha_k|\right) \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus we have proved (5.3).

(c) The characterisation of the class (c, c) is an immediate consequence of Part (a), and Theorems 3.17 and 3.18.
Now we assume $A \in (c, c)$. Let $x \in c$ be given and $\xi = \lim_{k\to\infty} x_k$. Then $x - \xi e \in c_0$ and it follows from (5.3) and (5.5) that

$$A_n x = A_n(x - \xi e) + \xi A_n e \to \sum_{k=0}^{\infty} \alpha_k (x_k - \xi) + \xi \alpha = \xi(\alpha - \sum_{k=0}^{\infty} a_k) + \sum_{k=0}^{\infty} \alpha_k x_k,$$

which is (5.5).
Finally, the characterisation of the class $(c, c; P)$ is an immediate consequence of the characterisation of (c, c) and (5.5). □

Example **5.5.** *We have* $(\ell_1, \ell_1) = \mathcal{B}(\ell_1, \ell_1)$ *and* $A \in (\ell_1, \ell_1)$ *if and only if*

$$\|A\|_{(1,1)} = \sup_k \sum_{n=0}^{\infty} |a_{nk}| < \infty. \tag{5.7}$$

If $A \in (\ell_1, \ell_1)$ *then*

$$\|L_A\| = \|A\|_{(\ell_1,\ell_1)}. \tag{5.8}$$

Proof. Since ℓ_1 has AK, Theorem 3.20 (b) yields the first part.
We apply the second part of Theorem 5.2 with $X = \ell_1$, $Z = c_0$, BK spaces with AK, and $Y = Z^\beta = \ell_1$ to obtain $A \in (\ell_1, \ell_1)$ if and only if $A^T \in (\ell_\infty, \ell_\infty)$; by Example 5.4 (a), this is the case if and only if (5.7) is satisfied.
Furthermore, if $A \in (\ell_1, \ell_1)$ then

$$\|L_A(x)\|_1 = \sum_{n=0}^{\infty} \left| \sum_{k=0}^{\infty} a_{nk} x_k \right| \le \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} |a_{nk} x_k| \le \|A\|_{(1,1)} \|x\|_1$$

implies $\|L_A\| \le \|A\|_{(1,1)}$. Also $L_A \in \mathcal{B}(\ell_1, \ell_1)$ implies $\|L_A(x)\|_1 = \|Ax\|_1 \le \|L_A\| \, \|x\|_1$, and it follows from $\|e^{(k)}\|_1 = 1$ for all k that $\|A\|_{(1,1)} = \sup_k \sum_{n=0}^{\infty} |a_{nk}| = \sup_k \|L_A(e^{(k)})\|_1 \le \|L_A\|$. □

6. Measures of Noncompactness

Now we find necessary and sufficient conditions for a matrix $A \in (X, Y)$ to define a compact operator L_A. This can be achieved by applying the *Hausdorff measure of noncompactness.*

The first measure of noncompactness was defined and studied by Kuratowski ([Kur]), and later used by Darbo ([Dar]). The Hausdorff measure of noncompactness was introduced and studied by Goldenstein, Gohberg and Markus ([GGM]). Istrăţesku introduced and studied the Istrăţesku measure of noncompactness ([Ist]). The interested reader is referred for measures on noncompactness to [AKP, B–G, Ist1, TBA, M–R].

We only consider the Hausdorff measure of noncompactness; it is the most suitable one for our purposes.

We recall of few standard notations and definitions.

The convex hull of a set S in a linear space X is the set

$$\operatorname{conv}(S) = \left\{ x = \sum_{k=0}^{n} \lambda_k s_k : \lambda_k > 0, s_k \in S \ (k = 0, 1, \dots, n) \text{ and } \sum_{k=0}^{n} \lambda_k = 1 \right\}$$

of (finite) convex linear combinations of S.

Let (X, d) be a metric space, $x_0 \in X$ and $r > 0$. By $B(x_0, r) = \{x \in X : d(x, x_0) < r\}$ we denote the open ball of radius r, centred at x_0. If M is a subset of X, then $\bar{M}$ denotes the closure of M. A set in a metric space is said to be *totally bounded*, if for every $\varepsilon > 0$ it can be covered by a finite number of open balls of radius ε. It is well–known that a subset M of a metric space is compact if and only if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of the subsequence is in M. The set M is said to be *relatively compact* if the closure $\bar{M}$ of M is a compact set. If the set M is relatively compact, then M is totally bounded. If the metric space is complete, then the set M is relatively compact if and only if it is totally bounded. It is easy to prove that a subset M of a metric space X is relatively compact if and only if every sequence (x_n) in M has a convergent subsequence; in this case, the limit of the subsequence need not be in M.

Let X and Y be infinite–dimensional complex Banach spaces. A linear operator L from X to Y is called *compact* if the domain of L is all of X, and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y.

Now we give the definition of the Hausdorff measure of compactness of bounded sets in a metric space.

Definition 6.1. *Let (X, d) be a metric space and $\mathcal{M}$ denote the collection of bounded subsets of X. The function $\chi : \mathcal{M} \to [0, \infty)$ with*

$$\chi(Q) = \inf \left\{ \varepsilon > 0 : Q \subset \bigcup_{k=0}^{n} B(x_k, r_k);\ x_k \in X,\ r_k < \varepsilon \ (n = 0, 1, 2 \dots) \right\}$$

is called Hausdorff measure of noncompactness*; $\chi(Q)$ is called the* Hausdorff measure of noncompactness of Q*.*

The Hausdorff measure of noncompactness has the following basic properties.

Proposition 6.2. ([M–R, Lemma 2.11, p. 168]) *Let X be a metric space and Q,Q_1,Q_2 $\in \mathcal{M}$. Then we have*

(i) $\chi(Q) = 0$ *if and only if Q is totally bounded,*

(ii) $\chi(Q) = \chi(\bar{Q})$,

(iii) $Q_1 \subset Q_2$ *implies* $\chi(Q_1) \le \chi(Q_2)$,

(iv) $\chi(Q_1 \cup Q_2) = \max\{\chi(Q_1), \chi(Q_2)\}$,

(v) $\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}$.

Proof. (i), (iii) The statements in (i) and (iii) follow directly from Definition 6.1.

(ii) We have $\chi(Q) \le \chi(\bar{Q})$ by (iii).
Let $\rho = \chi(Q)$. Then, given $\varepsilon > 0$, there are $n = n(\varepsilon) \in \mathbb{N}$ and $x_k \in X$ such that $Q \subset \bigcup_{k=0}^{n} B(x_k, \rho + \varepsilon/2)$, and it follows that

$$\bar{Q} \subset \overline{\bigcup_{k=0}^{n} B(x_k, \rho + \varepsilon/2)} = \bigcup_{k=0}^{n} \overline{B(x_k, \rho + \varepsilon/2)} \subset \bigcup_{k=0}^{n} B(x_0, \rho + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, this implies $\chi(\bar{Q}) \le \rho$.
(iv) It follows from (iii), that $\chi(Q_j) \le \chi(Q_1 \cup Q_2)$ for $j = 1, 2$, hence

$$\max\{\chi(Q_1), \chi(Q_2)\} \le \chi(Q_1 \cup Q_2). \tag{6.1}$$

Now let $\rho = \max\{\chi(Q_1), \chi(Q_2)\}$ and $\varepsilon > 0$ be given. Then, by Definition 6.1, Q_1 and Q_2 can be covered by finite unions of open balls of radius $\rho + \varepsilon$. Obviously the union of these covers is a finite cover of $Q_1 \cup Q_2$. This implies $\chi(Q_1 \cup Q_2) \le \rho + \varepsilon$, and since $\varepsilon > 0$ was arbitrary, it follows that $\chi(Q_1 \cup \chi(Q_2) \le \rho$. Now this and (6.1) together imply the equality in (iv).
(v) It follows from (iii) that $\chi(Q_1 \cap Q_2) \le \chi(Q_j)$ for $j = 1, 2$, hence $\chi(Q_1 \cap Q_2) \le \min\{\chi(Q_1), \chi(Q_2)\}$. □

Proposition 6.3. ([M–R, Theorem 2.12, p. 169]) *Let X be a normed space and Q,Q_1,Q_2 $\in \mathcal{M}$. Then we have*

(i) $$\chi(Q_1 + Q_2) \le \chi(Q_1) + \chi(Q_2),$$

(ii) $$\chi(Q + x) = \chi(Q) \text{ for all } x \in X,$$

(iii) $$\chi(\lambda Q) = |\lambda|\chi(Q) \text{ for all scalars},$$

(iv) $$\chi(Q) = \chi(\mathrm{conv}(Q)).$$

Proof. We denote the norm of X by $\|\cdot\|$.
(i) Let $\rho_j = \chi(Q_j)$ for $j = 1, 2$, $\rho = \rho_1 + \rho_2$, and $\varepsilon > 0$ be given. Then there are $n_j = n_j(\varepsilon) \in \mathbb{N}_0$ and $x_k^{(j)} \in X$ $(0 \le k \le n_j)$ for $j = 1, 2$ such that

$$Q_j \subset \bigcup_{k=0}^{n_j} B(x_k^{(j)}, \rho_j + \varepsilon/2) \text{ for } j = 1, 2. \tag{6.2}$$

Let $x \in Q_1 + Q_2$. Then there are $x_j \in Q_j$ $(j = 1, 2)$ such that $x = x_1 + x_2$, and it follows from (6.2) that there are $k_j \in \{0, 1, \dots, n_j\}$ such that $x_j \in B(x_{k_j}^{(j)}, \rho_j + \varepsilon/2)$ for $j = 1, 2$. This implies $\|x - (x_{k_1}^{(1)} + x_{k_2}^{(2)})\| \le \|x_1 - x_{k_j}^{(1)}\| + \|x_2 - x_{k_2}^{(2)}\| < \rho + \varepsilon$, and so $Q_1 + Q_2 \subset \bigcup_{k=0}^{n_1} \bigcup_{j=0}^{n_2} B(x_k^{(1)} + x_j^{(2)}, \rho + \varepsilon)$. Since $\varepsilon > 0$ was arbitrary, we conclude $\chi(Q_1 + Q_2) \le \rho$.
(ii) Let $x \in X$. Since obviously $\chi(\{x\}) = \chi(\{-x\}) = 0$, it follows from (i) that

$$\begin{aligned}\chi(Q) &= \chi\left((Q + x) - x\right) \le \chi(Q + x) + \chi(\{-x\}) \\ &= \chi(Q + x) \le \chi(Q) + \chi(\{x\}) = \chi(Q).\end{aligned}$$

(iii) Since the equality in (iii) is trivial for $\lambda = 0$, we assume $\lambda \neq 0$. Let $\rho = \chi(Q)$ and $\varepsilon > 0$ be given. Then we have

$$Q \subset \bigcup_{k=0}^{n} B(x_k, \rho + \varepsilon) \tag{6.3}$$

Let $y \in \lambda Q$ be given. Then there are $x \in Q$ such that $y = \lambda x$, and $k_0 \in \{0, 1, \dots, n\}$ such that $x \in B(x_k, \rho + \varepsilon)$. We put $y_k = \lambda x_k$ for $k = 0, 1, \dots$ and obtain $\|y - y_{k_0}\| = |\lambda| \, \|x - x_k\| < |\lambda|(\rho + \varepsilon)$. This implies $\lambda Q \subset \bigcup_{k=0}^{\infty} B(y_k, |\lambda|(\rho + \varepsilon))$. Since $\varepsilon > 0$ was arbitrary, we conclude $\chi(\lambda Q) \leq |\lambda|\rho = |\lambda|\chi(Q)$. Furthermore, it follows by what we have just shown that $\chi(Q) = \chi(\lambda^{-1}(\lambda Q)) \leq |\lambda^{-1}|\chi(\lambda Q)$, hence $|\lambda|\chi(Q) \leq \chi(\lambda Q)$.

(iv) Since obviously $Q \subset \text{conv}(Q)$, we obtain $\chi(Q) \leq \chi(\text{conv}(Q))$.
We have to show

$$\chi(\text{conv}(Q)) \leq \chi(Q). \tag{6.4}$$

Let $\rho = \chi(Q)$ and $\varepsilon > 0$ be given. Then we have (6.3), and every ball $B_k = B(x_k, \rho + \varepsilon)$ is a convex set. To see this, let $x, y \in B_k$ and $0 \leq \lambda \leq 1$. Then we have

$$\|\lambda x + (1 - \lambda)y - x_k\| \leq \|\lambda(x - x_k)\| + \|(1 - \lambda)(y - x_k)\| < (\lambda + (1 - \lambda))(\rho + \varepsilon) < \rho + \varepsilon.$$

We define $\sigma = \{\lambda = (\lambda_0, \dots, \lambda_n) \in \mathbb{R}^n : \sum_{k=0}^{n} \lambda_k = 1 \text{ and } \lambda_k \geq 0 \text{ for } k = 0, \dots, n\}$ and $A(\lambda) = \sum_{k=0}^{n} \lambda_k B_k$ for every $\lambda \in \sigma$.
It follows from (i) and (iii) that

$$\chi(A(\lambda)) \leq \sum_{k=0}^{n} \lambda_k \chi(B_k) \leq \rho + \varepsilon. \tag{6.5}$$

Now we show that the set $A = \bigcup_{\lambda \in \sigma} A(\lambda)$ is convex.
Let $x, y \in A$. Then there are $\lambda, \mu \in \sigma$ such that $x \in A(\lambda)$ and $y \in A(\mu)$, hence $x = \sum_{k=0}^{n} \lambda_k x_k$ and $y = \sum_{k=0}^{n} \mu_k y_k$ with $\lambda = (\lambda_0, \dots, \lambda_n), \mu = (\mu_0, \dots, \mu_n)$ and $x_k, y_k \in B_k$ $(k = 0, 1, \dots)$. We put $z = tx + (1 - t)y$ where $0 \leq t \leq 1$ and $\eta = t\lambda + (1 - t)\mu$ and have to show $z \in A(\eta)$ for some $\eta \in \sigma$. Putting $\eta_k = t\lambda_k + (1 - t)\mu_k$, $\xi_k = t\lambda_k/\eta_k$ for $\eta_k > 0$ and $\xi_k = 0$ for $\eta_k = 0$, and $z_k = \xi_k x_k + (1 - \xi)y_k$ $(k = 0, 1, \dots, n)$, we obtain

$$\sum_{k=0}^{n} \eta_k z_k = \sum_{k=0}^{n} (\xi_k x_k + (1 - \xi)y_k) = \sum_{k=0}^{n} (t\lambda_k x_k + (1 - t)\mu_k y_k) = z.$$

Since each B_k is a convex set, we have $z_k \in B_k$ for $k = 0, 1, \dots$. Furthermore, we obviously have $\eta_k \geq 0$ and $\sum_{k=0}^{n} \eta_k = t \sum_{k=0}^{n} \lambda_k + (1 - t) \sum_{k=0}^{n} \mu_k = 1$, hence $\eta \in \sigma$ and so $z \in A(\eta)$. Thus we have shown that A is convex.
Now we can prove the result.
Since $Q \subset \bigcup_{k=0}^{n} B_k \subset A$ and the set A is convex, it follows that $\text{conv}(B) \subset A$. Since the set σ is compact, given $\varepsilon > 0$, we can find finitely many $\lambda^{(0)}, \dots, \lambda^{(m)} \in \sigma$ such that for all $\lambda \in \sigma$ we have $\min_{k=0,\dots,m}\{\|\lambda - \lambda^{(k)}\|_1\} < \varepsilon/M$ where $M = \sup_{k=0,\dots,n}\{\|x\| : x \in B_k\} < \infty$. So if $x \in A$, $x = \sum_{k=0}^{n} \lambda_k x_k$, $\lambda_k \geq 0$, $\sum_{k=0}^{n} \lambda_k = 1$, then there exists $j \in \{0, 1, \dots, m\}$ such that $\sum_{k=0}^{n} |\lambda_k - \lambda_k^{(j)}| < \varepsilon/M$. We put $\bar{x} = \sum_{k=0}^{n} \lambda_k^{(j)} x_k$ and obtain

$\|x-\bar{x}\| \leq \sum_{k=0}^{n} |\lambda_k - \lambda_k^{(j)}| \, \|x_k\| < \varepsilon$, and therefore $\operatorname{conv}(B) \subset \bigcup_{j=0}^{m} A(\lambda^{(j)}) + \varepsilon \bar{B}(0,1)$. Thus we have by Proposition 6.2 (iv) and (6.5)

$$\chi(\operatorname{conv}(B)) \leq \max_{j=0,\dots,m} \{\chi(A(\lambda^{(j)}) + \chi(\varepsilon(\bar{B}(0,1))\} \leq \rho + \varepsilon + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that (6.4) holds. □

Theorem 6.4. ([M–R, Theorem 2.13, p. 169]) *Let X be an infinite–dimensional normed space. Then $\chi(\bar{B}(0,1)) = 1$.*

Proof. We write $B = \bar{B}(0,1)$.
Obviously we have $\chi(B) \leq 1$. If $\chi(B) = \rho < 1$ then we choose $\varepsilon > 0$ such that $\rho + \varepsilon < 1$. Then we have $B \subset \bigcup_{k=0}^{n} B(x_k, \rho+\varepsilon) \subset \bigcup_{k=0}^{n}(x_k + (\rho+\varepsilon)B)$, and it follows from Proposition 6.2 (iv) and Proposition 6.3 (ii) and (iii) that

$$\rho = \chi(B) \leq \max_{0 \leq k \leq n} \chi\left(x_k + (q+\varepsilon)B\right) = (\rho+\varepsilon)q.$$

Since $q + \varepsilon < 1$, this implies $q = 0$, and so B is a totally bounded set by Proposition 6.2 (i). But this is impossible, since X is an infinite–dimensional space. Thus we have $\chi(B) = 1$. □

Theorem 6.5 (Goldenstein, Gohberg, Markus)**.** ([GGM]; [M–R, Theorem 2.23, p. 173]) *Let X be a Banach space with a Schauder basis $(b_k)_{k=0}^{\infty}$, $Q \in \mathcal{M}$ and $\mathcal{P}_n : X \to X$ be the projector onto the linear span of $\{b_0, b_1, \dots, b_n\}$. Then we have*

$$\frac{1}{a} \limsup_{n\to\infty} \left(\sup_{x \in Q} \|(I - \mathcal{P}_n)(x)\| \right) \leq \chi(Q) \leq \limsup_{n\to\infty} \left(\sup_{x \in Q} \|(I - \mathcal{P}_n)(x)\| \right) \tag{6.6}$$

where $a = \limsup_{n\to\infty} \|I - \mathcal{P}_n\|$.

Proof. Obviously we have for every non–negative integer n

$$Q \subset \mathcal{P}_n(Q) + (I - \mathcal{P}_n)(Q). \tag{6.7}$$

It follows from (6.7) and Propositions 6.2 and 6.3 that

$$\chi(Q) \leq \chi(\mathcal{P}_n(Q)) + \chi((I - \mathcal{P}_n)(Q)) = \chi((I - \mathcal{P}_n)(Q)) \leq \sup_{x \in Q} \|(I - \mathcal{P}_n)(x)\|, \tag{6.8}$$

and we obtain

$$\chi(Q) \leq \inf_n \left(\sup_{x \in Q} \|(I - \mathcal{P}_n)(x)\| \right) \leq \limsup_{n\to\infty} \left(\sup_{x \in Q} \|(I - \mathcal{P}_n)(x)\| \right). \tag{6.9}$$

This proves the second inequality in (6.6).
Now we show the first inequality in (6.6).
Let $\rho = \chi(Q)$ and $\varepsilon > 0$ be given. Then we have

$$Q \subset \bigcup_{k=0}^{n} B_k(x_k, \rho+\varepsilon) \subset \{x_0, x_1, \dots, x_n\} + (\rho+\varepsilon)\bar{B}(0,1).$$

This implies that for every $x \in Q$ there exist $y \in \{x_1, x_2, \dots, x_n\}$ and $z \in \bar{B}(0,1)$ such that $x = y + (\rho+\varepsilon)z$, and so

$$\sup_{x\in Q} \|(I-\mathcal{P}_n)(x)\| \le \sup_{0\le k\le n} \|(I-\mathcal{P}_n)(x_k)\| + (\rho+\varepsilon)\|I-\mathcal{P}_n\|.$$

This yields

$$\limsup_{n\to\infty}\left(\sup_{x\in Q}\|(I-\mathcal{P}_n)(x)\|\right) \le (\rho+\varepsilon)\limsup_{n\to\infty}\|I-\mathcal{P}_n\|.$$

Since $\varepsilon > 0$ was arbitrary, the first inequality in (6.6) follows. □

So far we considered the measure of noncompactness of bounded subsets of a metric space. Now we define the *measure of noncompactness of an operator*.

Definition 6.6. *Let κ_1 and κ_2 be measures of noncompactness on the Banach spaces X and Y, and $\mathcal{M}_X$ and $\mathcal{M}_Y$ denote the collections of bounded sets in X and Y. An operator $L : X \to Y$ is said to be (κ_1,κ_2)–*bounded *if*

$$L(Q) \in \mathcal{M}_Y \text{ for all } Q \in \mathcal{M}_X$$

and there exists a non–negative real c such that

$$\kappa_2(L(Q)) \le c\,\kappa_1(Q) \text{ for all } Q \in \mathcal{M}_X.$$

If an operator L is (κ_1,κ_2)–bounded, then the number

$$\|L\|_{(\kappa_1,\kappa_2)} = \inf\{c \ge 0 : \kappa_2(L(Q)) \le c\,\kappa_1(Q) \text{ for all } Q \in \mathcal{M}_X\} \tag{6.10}$$

*is called the (κ_1,κ_2)–*measure of noncompactness *of L.*
If $\kappa = \kappa_1 = \kappa_2$, then we write $\|L\|_\kappa = \|L\|_{(\kappa,\kappa)}$.

The following result is useful to compute the Hausdorff measure of noncompactness of a bounded linear operator between Banach spaces.

Theorem 6.7. ([M–R, Theorem 2.25, p. 175]) *Let X and Y be Banach spaces, $L \in \mathcal{B}(X,Y)$, S_X and $\bar{B}_X$ be the unit sphere and the closed unit ball in X, and χ be the Hausdorff measure of noncompactness. Then we have*

$$\|L\|_\chi = \chi(L(S_X)) = \chi(L(\bar{B}_X)). \tag{6.11}$$

Proof. Since $\operatorname{conv}(S) = \bar{B}_X$ and $L(\operatorname{conv}(S_X)) = \operatorname{conv}(L(S_X))$, it follows from Proposition 6.3 (iv) that

$$\chi(L(\bar{B}_X)) = \chi(L(\operatorname{conv}(S_X))) = \chi(\operatorname{conv}(L(S_X))) = \chi(L(S_X)), \tag{6.12}$$

and we have $\chi(L(\bar{B}_X)) \le \|L\|_\chi\,\chi(\bar{B}) = \|L\|_\chi$ by (6.10) and Theorem 6.4.
To prove the converse inequality, let $Q \in \mathcal{M}_X$ be given, $\rho = \chi(Q)$ and $\varepsilon > 0$ be given. Then we have $Q \subset \bigcup_{k=0}^n B(x_k, \rho+\varepsilon)$ and obviously $L(Q) \subset \bigcup_{k=0}^n L(B(x_k,\rho+\varepsilon)$. It

follows from this, Proposition 6.2 (ii), Proposition 6.2 (ii),and (iii) that

$$\begin{aligned}\chi(L(Q)) &\le \chi\left(\bigcup_{k=0}^{n} L(B(x_k,\rho+\varepsilon))\right) \le \chi\left(L(\{x_0,\dots,x_n\}+B(0,\rho+\varepsilon)\right)\\ &= \chi(L(B(0,\rho+\varepsilon))) \le \chi(L((\rho+\varepsilon)B(0,1))) = \chi((\rho+\varepsilon)L(B(0,1)))\\ &\le (\rho+\varepsilon)\chi(L(\bar{B})).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $\chi(L(Q)) \le \rho\chi(L(\bar{B}_X)) = \chi(Q)\chi(L(\bar{B}_X))$ for all $Q \in \mathcal{M}_X$, hence $\|L\|_\chi \le \chi(L(\bar{B}_X))$. □

Theorem 6.8. ([M–R, Corollary 2.26, p. 175]) *Let X, Y and Z be Banach spaces, $L \in \mathcal{B}(X,Y)$, $\tilde{L} \in \mathcal{B}(Y,Z)$, and $\mathcal{C}(X,Y)$ denote the set of compact operators in $\mathcal{B}(X,Y)$. Then $\|\cdot\|_\chi$ is a seminorm on $\mathcal{B}(X,Y)$ and*

$$\begin{aligned}\|L\|_\chi &= 0 \text{ if and only if } L \in \mathcal{C}(X,Y),\\ \|L\|_\chi &\le \|L\|,\\ \|\tilde{L}\circ L\|_\chi &\le \|\tilde{L}\|_\chi\,\|L\|_\chi.\end{aligned} \tag{6.13}$$

Proof. First we show that $\|L\|_\chi$ is a seminorm.
Obviously we have $\|L\|_\chi \in [0,\infty)$ for all $L \in \mathcal{B}(X,Y)$.
Now we show $\|\lambda L\|_\chi = |\lambda|\,\|L\|_\chi$ for all scalars λ and all $L \in \mathcal{B}(X,Y)$. Since trivially $\|\lambda L\|_\chi = |\lambda|\,\|L\|_\chi$ for $\lambda = 0$, we may assume that $\lambda \ne 0$. Let $\rho = \|L\|_\chi$ and $\varepsilon > 0$ be given. Then we have $\chi(L(Q)) \le (\rho+\varepsilon)\chi(Q)$ and so $\chi(\lambda L(Q)) = |\lambda|\,\chi(L(Q)) \le |\lambda|(\rho+\varepsilon)\chi(Q)$ for all $Q \in \mathcal{M}_X$. Since $\varepsilon > 0$ was arbitrary, it follows that $\chi(\lambda L(Q)) \le |\lambda|\,\chi(Q)$ for all $Q \in \mathcal{M}_X$, hence $\|\lambda L\|_\chi \le |\lambda|\rho = |\lambda|\,\|L\|_\chi$. Also it follows by what we have just shown that $\|L\|_\chi = \|\lambda^{-1}(\lambda L)\|_\chi \le |\lambda^{-1}|\,\|\lambda L\|_\chi$, and so $|\lambda|\,\|L\|_\chi \le \|\lambda L\|_\chi$.
Now we prove the triangle inequality.
Let $\rho_k = \|L_k\|_\chi$ for $k = 1,2$, and $\varepsilon > 0$ be given. Then we have $\chi(L_k(Q)) \le (\rho_k + \varepsilon/2)\chi(Q)$ for all $Q \in \mathcal{M}_X$, and so by Proposition 6.3 (i)

$$\chi\left((L_1+L_2)(Q)\right) = \chi\left((L_1(Q)+L_2(Q)\right) \le \chi(L_1(Q)) + \chi(L_2(Q)) \le (\rho_1+\rho_2+\varepsilon)\chi(Q)$$

for all $Q \in \mathcal{M}_X$. Since $\varepsilon > 0$ was arbitrary, this implies $\chi((L_1+L_2)(Q)) \le (\rho_1+\rho_2)\chi(Q)$ for all $Q \in \mathcal{M}_X$, hence $\|L_1+L_2\|_\chi \le \rho_1+\rho_2 = \|L_1\|_\chi + \|L_2\|_\chi$ which is the triangle inequality.
Thus we have shown that $\|\cdot\|_\chi$ is a seminorm.

The statement in (6.13) is trivial in view of the remarks at the beginning of this section.

(i) Let $\bar{B}_X$ and $\bar{B}_Y$ denote the closed unit balls in X and Y. If $y \in L(\bar{B}_X)$ then there is $x \in \bar{B}_X$ such that $y = L(x)$ and $\|y\| = \|L(x)\| \le \|L\|$, hence $L(\bar{B}_X) \subset \overline{B_Y(0,\|L\|)} \subset \|L\|\,\bar{B}_Y$, and it follows from (6.11) in Theorem 6.7, Propositions 6.2 (iii) and 6.3 (iii), and Theorem (6.4) that $\|L\|_\chi = \chi(L(\bar{B}_X)) \le \chi(\|L\|\,\bar{B}_Y) = \|L\|\,\chi(\bar{B}_Y) = \|L\|$.

(ii) Let $Q \in \mathcal{M}_X$ be given. Then we have $\chi((\tilde{L}\circ L)(Q)) = \chi(\tilde{L}(L(Q))) \le \|\tilde{L}\|_\chi\,\chi(L(Q)) \le \|\tilde{L}\|_\chi\,\|L\|_\chi\,\chi(Q)$, and (ii) follows from Definition 6.6. □

Now we apply our results to characterise the classes $\mathcal{C}(\ell_1, \ell_1)$ and $\mathcal{C}(c, c)$. First we characterise the class $\mathcal{C}(\ell_1, \ell_1)$.

Theorem 6.9. ([M–R, Theorem 2.27, p. 175]) *Let $L \in \mathcal{B}(\ell_1, \ell_1)$, and A denote the infinite matrix such that $L(x) = Ax$ for all $x \in \ell_1$. Then we have $L \in \mathcal{C}(\ell_1, \ell_1)$ if and only if*

$$\lim_{r \to \infty} \left(\sup_k \sum_{n=r}^{\infty} |a_{nk}| \right) = 0. \tag{6.14}$$

Proof. By Theorem 3.20 (b), every $L \in \mathcal{B}(X, Y)$ can be represented by a matrix $A \in (X, Y)$. Writing $S = S_{\ell_1}$, we have $\|L\|_\chi = \chi(L(S))$ by (6.11) in Theorem 6.7. For $r = 0, 1, \dots$, let $A^{(r)}$ be the matrix with the first r rows replaced by 0. Then we obtain $\|(I - \mathcal{P}_{r-1})(L(x))\|_1 = \|A^{(r)}x\|_1$ hence, by (5.7) in Example 5.5,

$$\sup_{x \in S} \|(I - \mathcal{P}_{r-1})(L(x))\|_1 = \|A^{(r)}\|_{(\ell_1, \ell_1)} = \sup_k \sum_{n=r}^{\infty} |a_{nk}|.$$

Since obviously $\|I - \mathcal{P}_{r-1}\| = 1$ for all r, and the limit in (6.14) exists, it follows from (6.6) in Theorem 6.5 that $\chi(L(S)) = \lim_{r\to\infty}(\sup_k \sum_{n=r}^{\infty} |a_{nk}|)$. Finally it follows from (6.13) in Theorem 6.8 that $L \in \mathcal{C}(\ell_1, \ell_1)$ if and only if (6.14) is satisfied. □

Remark **6.10.** *It follows from Theorem 6.9 and Example 5.5 that every $L \in \mathcal{B}(\ell_1, \ell_1)$ is compact.*

Now we characterise the class $\mathcal{C}(X, Y)$.
First we give a representation of continuous linear operators from c to c.

Theorem 6.11. *We have $L \in \mathcal{B}(c, c)$ if and only if there exists a matrix $A \in (c_0, c)$ and a sequence $b \in \ell_\infty$ with*

$$\lim_{n \to \infty} \left(b_n + \sum_{k=0}^{\infty} a_{nk} \right) = \tilde{\alpha} \textit{ exists} \tag{6.15}$$

such that

$$L(x) = b \lim_{k \to \infty} x_k + Ax \textit{ for all } x \in c; \tag{6.16}$$

furthermore, we have

$$\|L\| = \sup_n \left(|b_n| + \sum_{k=0}^{\infty} |a_{nk}| \right). \tag{6.17}$$

Proof. First we assume that $L \in \mathcal{B}(c, c)$.
We write $L_n = P_n \circ L$ $(n = 0, 1, \dots)$ where P_n is the n–the coordinate with $P_n(x) = x_n$

($x \in \omega$). Since c is a BK space, we have $L_n \in c^*$ for all n, that is by (1.2)

$$L_n(x) = b_n \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty} a_{nk}x_k \ (x \in c) \tag{6.18}$$

$$\text{with } b_n = L_n(e) - \sum_{k=0}^{\infty} L_n(e^{(k)}) \text{ and } a_{nk} = L_n(e^{(k)}) \text{ for } k = 0, 1, \dots,$$

and by (1.3)

$$\|L_n\| = |b_n| + \sum_{k=0}^{\infty} |a_{nk}|. \tag{6.19}$$

Now (6.18) yields (6.16). Since $L(x_0) = Ax_0$ for all x_0, we have $A \in (c_0, c)$, and so $\|A\| = \sup_n \sum_{k=0}^{\infty}|a_{nk}| < \infty$ by (5.7) in Example 5.5. Also $L(e) = b + Ae$ implies (6.15), and we obtain $\|b\|_\infty \leq \|L(e)\|_\infty + \|A\| < \infty$, that is $b \in \ell_\infty$. Consequently we have $C = \sup_n(|b_n| + \sum_{k=0}^{\infty}|a_{nk}|) < \infty$. Now $\|L(x)\|_\infty = \sup_n |b_n \lim_{k\to\infty} x_k + \sum_{k=0}^{\infty}a_{nk}x_k| \leq (\sup_n(|b_n|+\sum_{k=0}^{\infty}|a_{nk}|)) \|x\|_\infty$ implies $\|L\| \leq C$. We also have $|L_n(x)| \leq \|L(x)\|_\infty \leq \|L\|$ for all $x \in \bar{B}_c$ and all n, and so $\sup_n \|L_n\| = C \leq \|L\|$. Thus we have shown (6.17).
Conversely we assume that $A \in (c_0, c)$ and $b \in \ell_\infty$ satisfy (6.15). Since $A \in (c_0, c)$ and $b \in \ell_\infty$, we obtain $C < \infty$ by (5.7) in Example 5.5, and so $L \in \mathcal{B}(c, \ell_\infty)$. Finally let $x \in c$ be given and $\xi = \lim_{k\to\infty} x_k$. Then we have $x - \xi e \in c_0$, $L_n(x) = b_n\xi + \sum_{k=0}^{\infty}a_{nk}x_k = (b_n + \sum_{k=0}^{\infty}a_{nk})\xi + A_n(x - \xi e)$ for all n, and it follows from (6.15) and $A \in (c_0, c)$ that $\lim_{n\to\infty} L_n(x)$ exists. Since $x \in c$ was arbitrary, we have $L \in \mathcal{B}(c, c)$. □

Now we apply Theorems 6.11 and 6.5.

Theorem 6.12. *Let $L \in \mathcal{B}(c, c)$. Using the notations of Theorem 6.11 and writing $\alpha_k = \lim_{n\to\infty} a_{nk}$ for all $k = 0, 1, \dots,$ we have*

$$\frac{1}{2} \limsup_{n\to\infty} \left(\left| b_n - \tilde{\alpha} + \sum_{k=0}^{\infty} \alpha_k \right| + \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) \leq \|L\|_\chi \tag{6.20}$$
$$\leq \limsup_{n\to\infty} \left(\left| b_n - \tilde{\alpha} + \sum_{k=0}^{\infty} \alpha_k \right| + \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right)$$

Proof. We assume that $L \in \mathcal{B}(c, c)$.
Let $x \in c$ be given, $\xi = \lim_{k\to\infty} x_k$ and $y = L(x)$. We have $y = b\xi + Ax$ where $A \in (c_0, c)$ and $b \in \ell_\infty$ by Theorem 6.11, and also note that the limits $\alpha_k = \lim_{n\to\infty} a_{nk}$ exist for all k by Example 5.4 (b). We can write

$$y_n = b_n\xi + A_nx = \xi \left(b_n + \sum_{k=0}^{\infty} a_{nk} \right) + A_n(x - \xi e) \text{ for all } . \tag{6.21}$$

Since $A \in (c_0, c)$ it follows from (5.3) in Example 5.2 that

$$\lim_{n\to\infty} A_n(x - \xi e) = \sum_{k=0}^{\infty} \alpha_k(x_k - \xi) = \sum_{k=0}^{\infty} \alpha_k x_k - \xi \sum_{k=0}^{\infty} \alpha_k. \tag{6.22}$$

Thus it follows from (6.21), (6.22) and (6.15) that

$$\eta = \lim_n y_n = \xi \left(\tilde{\alpha} - \sum_{k=0}^{\infty} \alpha_k \right) + \sum_{k=0}^{\infty} \alpha_k x_k. \tag{6.23}$$

We are going to apply the Goldenstein–Gohberg–Markus theorem (Theorem 6.5) to establish the estimate in (6.20).
First we note that $\|L\|_\chi = \chi(L(\bar{B}_c))$ by (6.11) in Theorem 6.7. Since every sequence $z = (z_k)_{k=0}^n \in c$ has a representation $z = \zeta e + \sum_{k=0}^{\infty}(z_k - \zeta)e^{(k)}$ with $\zeta = \lim_{k\to\infty}$, we define the projector $\mathcal{P}_r : c \to c$ by $\mathcal{P}_r(z) = \zeta e + \sum_{k=0}^{r}(z_k - \zeta)e^{(k)}$, and it follows that the sequence $\tilde{z} = (I - \mathcal{P})(z)$ is given by $\tilde{z}_k = 0$ for $0 \le k \le r$ and $\tilde{z}_k = z_k - \zeta$ for $k \ge r + 1$. Therefore we have $|\tilde{z}_k| \le |z_k| + |\zeta| \le 2\|z\|_\infty$ for all k, hence $\|I - \mathcal{P}_r\| \le 2$. Now let z be the sequence with $z_{r+1} = (-1)$ and $z_k = 1$ for $k \ne r + 1$. Then $\zeta = 1$, $\|z\|_\infty = 1$ and $\|(I - \mathcal{P}_r)(z)\|_\infty = 2$, hence $\|I - \mathcal{P}_r\| = 2$. Thus we it follows that

$$\lim_{r\to\infty} \|I - \mathcal{P}_r\| = 2. \tag{6.24}$$

Writing $f_n(x) = ((I - \mathcal{P}_r)(L(x)))_n$, we obtain for $n \ge r + 1$ by (6.21) and (6.23)

$$\begin{aligned} f_n(x) = y_n - \eta = \xi b_n + A_n(x) &- \left(\xi \left(\tilde{\alpha} - \sum_{k=0}^{\infty} \alpha_k \right) + \sum_{k=0}^{\infty} \alpha_k x_k \right) \\ = \xi \left(b_n - \tilde{\alpha} + \sum_{k=0}^{\infty} \alpha_k \right) &+ \sum_{k=0}^{\infty} (a_{nk} - \alpha_k) x_k, \end{aligned}$$

and see that $f_n \in c^*$ by (1.2), and $\|f_n\| = |b_n - \tilde{\alpha} + \sum_{k=0}^{\infty} \alpha_k| + \sum_{k=0}^{\infty} |a_{nk} - \alpha_k|$ by (1.3). Thus we have shown that

$$\sup_{x \in \bar{B}_c} \|(I - \mathcal{P}_r)(L(x))\| = \sup_{n \ge r+1} \|f_n\| = \sup_{n \ge r} \left(\left| b_n - \tilde{\alpha} + \sum_{k=0}^{\infty} \alpha_k \right| + \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right),$$

and (6.20) now follows from (6.24), (6.11) in Theorem 6.7 and (6.6) in Theorem 6.5. □

The characterisation of the class $\mathcal{C}(c, c)$ is an immediate consequence of Theorem 6.12.

Corollary 6.13. *Let $L \in \mathcal{B}(c, c)$. Then L is compact if and only if*

$$\lim_{n\to\infty} \left(\left| b_n - \tilde{\alpha} + \sum_{k=0}^{\infty} \alpha_k \right| + \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) = 0. \tag{6.25}$$

In particular, if $A \in (c, c)$ then L_A is compact if and only if

$$\lim_{n\to\infty} \left(\left| \sum_{k=0}^{\infty} \alpha_k - \tilde{a} \right| + \sum_{k=0}^{\infty} |a_{nk} - \alpha_k| \right) = 0. \tag{6.26}$$

Remark **6.14.** *It is obvious from the second part of Corollary 6.13 that if A is a regular matrix then L_A cannot be compact. If A is a conservative matrix and L_A is compact then A is conull, that is $\tilde{a} - \sum_{k=0}^{\infty} \alpha_k = 0$.*

An operator $L \in \mathcal{B}(c,c)$ is said to be regular if and only if $\lim_{n\to\infty}(L(x))_n = \lim_{k\to\infty} x_k$ for all $x \in c$.

Corollary 6.15 (Cohen–Dunford)**.** ([C–D, Corollary 3]) *Let $L \in \mathcal{B}(c,c)$ be regular. Then L is compact if and only if*

$$\lim_{n\to\infty}\left(|b_n - 1| + \sum_{k=0}^{\infty}|a_{nk}|\right) = 0. \tag{6.27}$$

Proof. We show that $L \in \mathcal{B}(c,c)$ is regular if and only if $\alpha_k = 0$ and $\tilde{\alpha} = 1$. Then the statement of the corollary is an immediate consequence of Corollary 6.13.
First we assume that $L \in \mathcal{B}(c,c)$ is regular. By Theorem 6.11 there are a matrix $A \in (c_0, c)$ such that (6.15) holds, and a sequence $b \in \ell_\infty$ such that $L(x) = b\lim_{k\to\infty} x_k + A(x)$ for all $x \in c$. Thus we have

$$\lim_{n\to\infty}(L(e^{(k)}))_n = 0 = \lim_{n\to\infty} a_{nk} = 0 \tag{6.28}$$

and

$$\lim_{n\to\infty}(L(e))_n = 1 = \lim_{n\to\infty}\left(b_n + \sum_{k=0}^{\infty} a_{nk}\right) = \tilde{\alpha}. \tag{6.29}$$

Conversely if $L(x) = b\lim_{k\to\infty} x_k + A(x)$ and (6.28) and (6.29) are satisfied then it follows from (6.23) that $\lim_{n\to\infty}(L(x))_n = \lim_{k\to\infty} x_k(\tilde{\alpha} - \sum_{k=0}^{\infty}\alpha_k) + \sum_{k=0}^{\infty}\alpha_k x_k = \lim_k x_k$ for all $x \in c$, and L is regular. □

Appendix A. Known Results from Functional Analysis

The following results are well known in functional analysis.

Theorem A.1 (The closed graph lemma)**.** ([Wil1, Theorem 11.1.1, p. 195]) *Any continuous map into a Hausdorff space has closed graph.*

Theorem A.2 (The closed graph theorem)**.** ([Wil1, Theorem 11.2.2, p. 200]) *If X and Y are Fréchet spaces and $f : X \to Y$ is a closed linear map, then f is continuous.*

Theorem A.3 (The Banach–Steinhaus theorem)**.** ([Wil1, Corollary 11.2.4, p. 200]) *Let (f_n) be a pointwise convergent sequence of linear functionals on a Fréchet space X. Then f defined by $f(x) = \lim_{n\to\infty} f(x)$ is continuous.*

Theorem A.4 (The Hahn–Banach theorem)**.** ([Wil2, 3.0.4, p. 39]) *Let X be a subspace of a linear topological space Y and f be a linear functional on X which is continuous in the relative topology of Y. Then f can be extended to a continuous linear functional on Y.*

Theorem A.5 (The uniform boundedness principle)**.** ([Wil1, Corollary 11.2.3, p. 200]) *Let (f_n) be a pointwise convergent sequence of continuous linear functionals on a Fréchet space. Then (f_n) is equicontinuous.*

Theorem A.6 (The convergence lemma)**.** ([Wil2, 7.0.3, p.103]) *Let* (f_n) *be a sequence of equicontinuous linear functionals on a linear topological space* X*. Then the set* $\{x \in X : \lim_{n\to\infty} f_n(x)$ *exists* $\}$ *is a closed linear subspace of* X*.*

Theorem A.7 (The open mapping theorem)**.** ([Wil1, Theorem 11.2.1, p. 199]) *Let* X *and* Y *be Fréchet spaces, and* $f : X \to Y$ *a closed linear map onto. Then* f *is open.*

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