Jordan Journal of Mathematics and Statistics (JJMS) 2 (1), 2009, pp. 43-54

A PREY-PREDATOR MODEL WITH COVER FOR THE PREY AND AN ALTERNATIVE FOOD FOR THE PREDATOR AND CONSTANT HARVESTING OF BOTH THE SPECIES *

K. LAKSHMI NARAYAN[1] N.PATTABHIRAMACHARYULU[2]

ABSTRACT

The present paper is devoted to an analytical investigation of a two species prey-predator model. Predator is provided with a limited resource of food in addition to the prey and a cover to prey proportionate to its population to get protection from the predator. Both the prey and predator are harvested at a constant rate. The model is characterized by couple of first order non-linear ordinary differential equations. The lone equilibrium point of the model is identified and its stability criteria is discussed. The Global stability of linearized equations is discussed by constructing a suitable Liapunov's function. A threshold theorem is stated and results are discussed.

1. INTRODUCTION

In the classical Lotka - Volterra Prey - Predator model, there is no protection for Prey from the Predator and Predator sustains on the Prey alone. When the Prey population falls below a certain level, the predator would migrate to another region in search of food and return only when the Prey population rises to the required level. Olinck,[1] gave an introduction to Mathematical modeling in life sciences. Kapur, [2], Smith,[3], Colinvaux, [4], Freedman, [5] discussed some of the prey-predator ecological models. May, [6] discussed stability and complexity of ecological models, Varma, [7] discussed about their exact solutions. Narayan & Ramacharyulu, [8], [9], [10], [11] and [12] discussed different prey-predator models.

2. Basic Equations

The model equations for a two species prey-predator system is given by a coupled non-linear ordinary differential equations employing the terminology given below.

N_1 and N_2 are the populations of the prey and predator with the natural growth rates a_1 and a_2 respectively,

α_{11} is rate of decrease of the prey due to insufficient food,

2000 **Mathematics Subject Classification:** 37C75.

Keywords: Equilibrium point, Normal Study State, Prey, Predator, Stability, Threshold Diagrams, and Threshold Theorems.

Received on: Dec. 4, 2008 **Accepted on**: June 16, 2009

α_{12} is rate of decrease of the prey due to inhibition by the predator,

α_{21} is rate of increase of the predator due to successful attacks on the prey,

α_{22} is rate of decrease of the predator due to insufficient food other than the prey,

h_1 is rate of harvest of the prey,

h_2 is rate of harvest of the predator,

and both h_1, h_2 are assumed to be positive constants.

(i)Equation for the growth rate of prey species (N_1):

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^{\,2} - \alpha_{12}(1-k) N_1 N_2 - h_1 . \qquad (2.1)$$

(ii)Equation for the growth rate of predator species (N_2):

$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^{\,2} + \alpha_{21}(1-k) N_1 N_2 - h_2 \qquad (2.2)$$

3. Equilibrium states

The equilibrium states for the system under investigation are given by

$\frac{dN_1}{dt} = 0$ and $\frac{dN_2}{dt} = 0$

i.e. $$N_1\{a_1 - \alpha_{11} N_1 - \alpha_{12}(1-k) N_2\} = h_1 \qquad (3.1)$$

$$N_2\{a_2 - \alpha_{22} N_2 + \alpha_{21}(1-k) N_1\} = h_2 \qquad (3.2)$$

(3.1) $\times \alpha_{21}(1-k) N_1$ + (3.2) $\times \alpha_{12}(1-k) N_2$ we get

$$h_1 \alpha_{21}(1-k) + h_2 \alpha_{12}(1-k) =$$

$$\alpha_{21}(1-k) a_1 N_1 - \alpha_{11}\alpha_{21}(1-k) N_1^{\,2} + \alpha_{12}(1-k) N_2 a_2 - \alpha_{22}\alpha_{12}(1-k) N_2^{\,2} \qquad (3.3)$$

On rearranging, the above terms can brought to the form

$$\alpha_{21}(1-k)\alpha_{11}\left(N_1 - \frac{a_1}{2\alpha_{11}}\right)^2 + \alpha_{12}(1-k)\alpha_{22}\left(N_2 - \frac{a_2}{2\alpha_{22}}\right)^2 + \alpha_{12}(1-k)\left(h_2 - \frac{a_2^{\,2}}{4\alpha_{22}}\right)$$

$$+\alpha_{21}(1-k)\left(h_1 - \frac{a_1^{\,2}}{4\alpha_{11}}\right) = 0 \ , \qquad (3.4)$$

which connects the harvesting rates and the normal steady state.

From this equation two cases can be drawn. They are

(i) Case of exclusive harvesting (i.e. the harvesting rates of prey and predator are independent of each other) is given as

$$h_1 = \frac{a_1^{\,2}}{4\alpha_{11}} \text{ and } h_2 = \frac{a_2^{\,2}}{4\alpha_{22}} \qquad (3.5)$$

(ii) Case of mixed or gross harvesting is characterized by

$$\alpha_{12}(1-k)\left(h_2 - \frac{a_2^{\,2}}{4\alpha_{22}}\right) + \alpha_{21}(1-k)\left(h_1 - \frac{a_1^{\,2}}{4\alpha_{11}}\right) = 0 \tag{3.6}$$

In either of the cases, the equilibrium values of N_1 and N_2 are related by

$$\alpha_{21}(1-k)\alpha_{11}\left(N_1 - \frac{a_1}{2\alpha_{11}}\right)^2 + \alpha_{12}(1-k)\alpha_{22}\left(N_2 - \frac{a_2}{2\alpha_{22}}\right)^2 = 0 \tag{3.7}$$

$\therefore$ The lone equilibrium point is

$\overline{N_1} = \dfrac{a_1}{2\alpha_{11}}$ (Half of the carrying capacity of N_1) and (3.8)

$\overline{N_2} = \dfrac{a_2}{2\alpha_{22}}$ (Half of the carrying capacity of N_2) (3.9)

4. Stability of Equilibrium State

Let *N* = (*N₁, N₂*) = $\overline{N} + U$ = $(\overline{N}_1 + u_1, \overline{N}_2 + u_2)$ (4.1)

where U = (u_1, u_2) is a small perturbation over the equilibrium state $\overline{N} = (\overline{N}_1, \overline{N}_2)$. The basic equations (2.1), (2.2) are quasi-linearized to obtain the equations for the perturbed state $\dfrac{dU}{dt} = AU$ where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}\overline{N}_1 - \alpha_{12}(1-k)\overline{N}_2 & -\alpha_{12}(1-k)\overline{N}_1 \\ \alpha_{21}(1-k)\overline{N}_2 & a_2 - 2\alpha_{22}\overline{N}_2 + \alpha_{21}(1-k)\overline{N}_1 \end{bmatrix} \tag{4.2}$$

The characteristic equation for the system is $det\left[A - \lambda I\right] = 0$ (4.3)

The equilibrium state is stable only when the roots of the equation (4.3) are negative, in case they are real or have negative real parts, in case they are complex.

Put $N_1 = u_1 + \overline{N_1}$ and $N_2 = u_2 + \overline{N_2}$ in (2.1) & (2.2), where u_1 and u_2 are small perturbations from the equilibrium state.

$$\frac{du_1}{dt} = (u_1 + \overline{N}_1)\{a_1 - \alpha_{11}(u_1 + \overline{N}_1) - \alpha_{12}(1-k)(u_2 + \overline{N}_2)\} - h_1 \tag{4.1}$$

$$\frac{du_2}{dt} = (u_2 + \overline{N}_2)\{a_2 - \alpha_{22}(u_2 + \overline{N}_2) - \alpha_{21}(1-k)(u_1 + \overline{N}_1)\} - h_2 \tag{4.2}$$

By neglecting products, higher powers of u_1, u_2, and constant terms we get

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N}_1 u_1 - \alpha_{12}(1-k)\overline{N}_1 u_2 \tag{4.3}$$

and $$\frac{du_2}{dt} = -\alpha_{22}\overline{N}_2 u_2 + \alpha_{12}(1-k)u_1\overline{N}_2 \quad (4.4)$$

The characteristic equation is

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)\lambda + [\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2]\ \overline{N}_1\overline{N}_2 = 0 \quad (4.5)$$

The roots of which can be noted to be negative.

∴ The co-existent equilibrium state is stable.

The trajectories are

$$u_1 = \left[\frac{u_{10}(\lambda_1 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1(1-k)}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t}$$

$$+ \left[\frac{u_{10}(\lambda_2 + \alpha_{22}\overline{N}_2) - u_{20}\alpha_{12}\overline{N}_1(1-k)}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \quad (4.6)$$

$$u_2 = \left[\frac{u_{20}(\lambda_1 + \alpha_{11}\overline{N}_1) - u_{10}\alpha_{21}\overline{N}_2(1-k)}{\lambda_1 - \lambda_2}\right] e^{\lambda_1 t} +$$

$$\left[\frac{u_{20}(\lambda_2 + \alpha_{11}\overline{N}_1) - u_{10}\alpha_{21}\overline{N}_2(1-k)}{\lambda_2 - \lambda_1}\right] e^{\lambda_2 t} \quad (4.7)$$

The curves are illustrated in **Figures 1& 2.**

CASE 1: Initially the prey dominates the predator and it continues through out its growth i.e. $u_{10} < u_{20}$. In this case the predator always outnumbers the prey. It is evident that both the species are asymptotic to the equilibrium point. Hence this state is stable.

CASE 2: The prey dominates the predator in natural growth rate but its initial strength is less than that of predator. i.e. $u_{10} > u_{20}$. In this case, initially the prey outnumber the predator and this continues the time instant $t = t^*$ given by equation (4.8), after which the predator out number the prey.

$$t = t^* = \frac{1}{\lambda_2 + \lambda_1}\ ln\left[\frac{(b_3 - a_5)u_{10} + (a_3 + b_1)u_{20}}{(b_2 - a_6)u_{10} + (a_4 + b_1)u_{20}}\right] \quad (4.8)$$

where $a_3 = \lambda_1 + \alpha_{11}\overline{N}_1$; $a_4 = \lambda_2 + \alpha_{11}\overline{N}_1$; $a_5 = \lambda_1 + \alpha_{22}\overline{N}_2$;

$a_6 = \lambda_2 + \alpha_{22}\overline{N}_2$ $b_1 = \alpha_{12}(1-k)\overline{N}_1$; $b_2 = \alpha_{21}(1-k)\overline{N}_2$. (4.9)

As $t \to \infty$ both u_1 & u_2 approaches the equilibrium point. Hence the state is stable.

4.2. Trajectories of perturbed species

The trajectories in the u_1-u_2 plane are given by

$$[u_2^{(1+a)(v_1+v_2)}]d = \frac{(u_1\text{-}u_2 v_1)^{(p_3\text{-}av_1)}}{(u_1 - v_2 u_2)^{(p_3 - av_1)}} \tag{4.10}$$

where, $p_3 = \overline{N}_2 \alpha_{22}$ (4.11)

and v_1, v_2 are roots of quadratic equation $av^2+bv+c_4=0$ with

$$a=\alpha_{21}\overline{N}_2(1-k)\,; \quad b=\alpha_{11}\overline{N}_1 - \alpha_{22}\overline{N}_2\,; \quad c_4=\alpha_{12}\overline{N}_1(1-k) \tag{4.12}$$

and d , an arbitrary constant.

When $(\alpha_{11}\overline{N}_1 - \alpha_{22}\overline{N}_2)^2 < 4\alpha_{12}\alpha_{21}(1-k)^2\overline{N}_1\overline{N}_2$, (4.13)

the roots are complex with negative real part. Hence the equilibrium state is **stable.** The solution curves are illustrated in **Figure .3.**

When $(\alpha_{11}\overline{N}_1 - \alpha_{22}\overline{N}_2)^2 > 4\alpha_{12}\alpha_{21}(1-k)^2\overline{N}_1\overline{N}_2$, (4.14)

the roots are real and negative. Hence the equilibrium state is **stable.**
The solution curves are illustrated in **Figure .4.**

5. Liapunov's Function for Global Stability

The Linearized Basic Equations for the model under investigation are

$$\frac{du_1}{dt} = -\alpha_{11}\overline{N}_1 u_1 - \alpha_{12}(1-k)\overline{N}_2 u_2 \tag{5.1}$$

$$\frac{du_2}{dt} = -\alpha_{21}(1-k)\overline{N}_2 u_1 - \alpha_{22}\overline{N}_2 u_2 \tag{5.2}$$

The characteristic equation is

$$(\lambda + \alpha_{11}\overline{N}_1)(\lambda + \alpha_{22}\overline{N}_2) + \alpha_{12}\alpha_{21}(1-k)^2\overline{N}_1\overline{N}_2 = 0 \;\Rightarrow \lambda^2 + p\lambda + q = 0 \tag{5.3}$$

where $p = \alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2 > 0$ and $q = \{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2\}\overline{N}_1\overline{N}_2 > 0$

Hence the conditions for Liapunov's function are satisfied.

Now we define $E(u_1,u_2) = \frac{1}{2}(au_2 + 2bu_1u_2 + cu_2^2)$ (5.4)

where $a = \dfrac{(\alpha_{21}(1-k)\overline{N}_2)^2 + (\alpha_{22}N_2)^2 + \{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2\}\overline{N}_1\overline{N}_2}{D}$ (5.5)

$$b = \frac{\alpha_{11}\alpha_{21}(1-k)\overline{N}_1\,\overline{N}_2 - \alpha_{12}\alpha_{22}(1-k)\overline{N}_1\,\overline{N}_2}{D} \tag{5.6}$$

$$c = \frac{(\alpha_{11}\overline{N}_1)^2 + (\alpha_{12}(1-k)\overline{N}_1)^2 + \{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2\}\overline{N}_1\,\overline{N}_2}{D} \text{ and} \tag{5.7}$$

$$D = pq = \{\alpha_{11}\overline{N}_1 + \alpha_{22}N_2\}\{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2\}\overline{N}_1\,\overline{N}_2 \tag{5.8}$$

From equations (5.5) & (5.8) it is clear that $D > 0$ and $a > 0$. Also

$$D^2(ac - b^2) =$$

$$D^2\Big\{\frac{(\alpha_{21}(1-k)\overline{N}_2)^2 + (\alpha_{22}N_2)^2 + \{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2\}\overline{N}_1\,\overline{N}_2}{D}$$

$$\times$$

$$\frac{(\alpha_{11}\overline{N}_1)^2 + (\alpha_{12}(1-k)\overline{N}_1)^2 + \{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}(1-k)^2\}\overline{N}_1\,\overline{N}_2}{D} -$$

$$\frac{\alpha_{11}^{\ 2}\alpha_{21}^{\ 2}(1-k)^2\overline{N}_1^{\ 2}\overline{N}_2^{\ 2} + \alpha_{12}^{\ 2}\alpha_{22}^{\ 2}(1-k)^2\overline{N}_1^{\ 2}\overline{N}_2^{\ 2} - 2\alpha_{11}\alpha_{12}\alpha_{21}\alpha_{22}(1-k)^2\overline{N}_1^{\ 2}\overline{N}_2^{\ 2}}{D^2}\Big\} > 0 \tag{5.9}$$

Since $D^2 > 0$, $ac - b^2$ is also greater than zero. (5.10)

$\therefore$ The function E(x, y) is positive definite.

Further,

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} = \begin{array}{l}(au_1 + bu_2)[-\alpha_{11}\overline{N}_1 u_1 - \alpha_{12}(1-k)\overline{N}_1 u_2] + \\ (bu_1 + cu_2)[\alpha_{21}(1-k)\overline{N}_2 u_1 - \alpha_{22}\overline{N}_2 u_2]\end{array}$$

$$= \begin{array}{l}(b\alpha_{21}(1-k)\overline{N}_2 - a\alpha_{11}\overline{N}_1)u_1^{\ 2} - (b\alpha_{12}(1-k)\overline{N}_1 + c\alpha_{22}\overline{N}_2)u_2^{\ 2} - \\ \{[b\alpha_{11} + a\alpha_{12}(1-k)]\overline{N}_1 + [b\alpha_{22} - c\alpha_{21}(1-k)\overline{N}_2\}u_1u_2\end{array} \tag{5.11}$$

By substituting the values of *a*, *b* and *c* from equations (5.5),(5.6) &(5.7) and on simplification, we get

$$\frac{\partial E}{\partial u_1}\frac{du_1}{dt} + \frac{\partial E}{\partial u_2}\frac{du_2}{dt} =$$

$$\Big\{\frac{[\alpha_{11}\alpha_{21}^{\ 2}(1-k)^2\overline{N}_1\,\overline{N}_2^{\ 2} - \alpha_{12}\alpha_{22}\alpha_{21}(1-k)\overline{N}_1\,\overline{N}_2^{\ 2} - \alpha_{11}\alpha_{21}^{\ 2}(1-k)^2\overline{N}_2^{\ 2}\overline{N}_1}{D} -$$

$$\frac{\alpha_{11}\alpha_{22}^{\ 2}\overline{N}_1^{\ 2}\overline{N}_2 - \alpha_{11}^{\ 2}\alpha_{22}\overline{N}_1^{\ 2}\overline{N}_2 + \alpha_{12}\alpha_{11}\alpha_{21}(1-k)^2\overline{N}_1^{\ 2}\overline{N}_2\,]}{D}\Big\}u_1^{\ 2} -$$

$$\Big\{\frac{[\alpha_{12}\alpha_{11}\alpha_{21}(1-k)^2\overline{N}_1^{\ 2}\overline{N}_2 - \alpha_{12}^{\ 2}\alpha_{22}(1-k)^2\overline{N}_1^{\ 2}\overline{N}_2 + \alpha_{11}^{\ 2}\alpha_{22}^{\ 2}\overline{N}_1^{\ 2}\overline{N}_2}{D} +$$

$$\frac{\alpha_{12}{}^2\alpha_{22}(1-k)^2\overline{N}_1{}^2\overline{N}_2+\alpha_{11}\alpha_{22}{}^2\overline{N}_2{}^2\overline{N}_1+\alpha_{12}\alpha_{22}\alpha_{21}\overline{N}_2{}^2\overline{N}_1]}{D}\}u_2{}^2-$$

$$\{\frac{\alpha_{11}{}^2\alpha_{21}(1-k)^2\overline{N}_1{}^2\overline{N}_2-\alpha_{12}\alpha_{11}\alpha_{22}(1-k)^2\overline{N}_1{}^2\overline{N}_2+\alpha_{12}\alpha_{21}{}^2(1-k)^3\overline{N}_2{}^2\overline{N}_1}{D}+$$

$$\frac{\alpha_{12}\alpha_{22}{}^2(1-k)\overline{N}_1\overline{N}_2{}^2+\alpha_{11}\alpha_{12}\alpha_{22}(1-k)\overline{N}_1{}^2\overline{N}_2+\alpha_{12}{}^2\alpha_{21}(1-k)^3\overline{N}_1{}^2\overline{N}_2}{D}-$$

$$\frac{\alpha_{12}\alpha_{22}{}^2(1-k)\overline{N}_1\overline{N}_2{}^2+\alpha_{11}\alpha_{12}\alpha_{22}(1-k)\overline{N}_1\overline{N}_2{}^2-\alpha_{11}{}^2\alpha_{21}(1-k)\overline{N}_1{}^2\overline{N}_2}{D}-$$

$$\frac{\alpha_{12}{}^2\alpha_{21}(1-k)^3\overline{N}_1{}^2\overline{N}_2-\alpha_{11}\alpha_{21}\alpha_{22}(1-k)\overline{N}_1\overline{N}_2{}^2-\alpha_{12}\alpha_{21}{}^2(1-k)^3\overline{N}_2{}^2\overline{N}_1}{D}\}u_1u_2=$$

$$-\frac{D}{D}u_1{}^2-\frac{D}{D}u_2{}^2$$

$$\Rightarrow\quad \frac{\partial E}{\partial u_1}\frac{du_1}{dt}+\frac{\partial E}{\partial u_2}\frac{du_2}{dt}=-(u_1{}^2+u_2{}^2)\,, \tag{5.12}$$

which is clearly negative definite. So E(x, y) is a Lyapunov's function for the linear system.

Further we prove that E (u_1, u_2) is also a Lyapunov's function for the non linear system.

If, F_1 and F_2 are defined by

$$F_1(N_1, N_2) = N_1\{a_1-\alpha_{11}N_1-\alpha_{12}(1-k)N_2\}-h_1 \tag{5.13}$$

$$F_2(N_1, N_2) = N_2\{a_2-\alpha_{22}N_2-\alpha_{21}(1-k)N_1\}-h_2\,, \tag{5.14}$$

we have to prove that $\frac{\partial E}{\partial u_1}F_1+\frac{\partial E}{\partial u_2}F_2$ is negative definite.

By substituting $N_1=\overline{N}_1+u_1$ and $N_2=\overline{N}_2+u_2$ in (5.1) & (5.2) equations, we get

$$\frac{du_1}{dt}= (\overline{N}_1+u_1)\{a_1-\alpha_{11}\overline{N}_1-\alpha_{11}u_1-\alpha_{12}(1-k)\overline{N}_2-\alpha_{12}(1-k)u_2\}-h_1$$

From (5.13), $F_1(u_1, u_2)=\frac{du_1}{dt}= -\alpha_{11}\overline{N}_1u_1-\alpha_{12}(1-k)\overline{N}_1u_2+f_1(u_1, u_2)$ (5.15)

where $f_1(u_1, u_2) = -\alpha_{11}u_1^2-\alpha_{12}(1-k)u_1u_2-h_1$ (5.16)

Similarly $F_2(u_1, u_2) = \frac{du_2}{dt} = -\alpha_{22}\overline{N}_2 u_2 - \alpha_{21}(1-k)\overline{N}_2 u_1 + f_2(u_1, u_2)$ (5.17)

where $f_2(u_1, u_2) = -\alpha_{22}u_2^{\ 2} + \alpha_{21}(1-k)u_1u_2 - h_2$ (5.18)

And we have $\frac{\partial E}{\partial u_1} = au_1 + bu_2$ and $\frac{\partial E}{\partial u_2} = bu_1 + cu_2$ (5.19)

By considering the equations (5.15),(5.17) and (5.19),

$$\frac{\partial E}{\partial u_1}F_1 + \frac{\partial E}{\partial u_2}F_2 = -(u_1^{\ 2} + u_2^{\ 2}) + (au_1 + bu_2)f_1(u_1, u_2) + (bu_1 + cu_2)f_2(u_1, u_2) \quad (5.20)$$

By introducing polar co-ordinates we get,

$$\frac{\partial E}{\partial u_1}F_1 + \frac{\partial E}{\partial u_2}F_2 =$$

$$-r^2 + r[(a\cos\theta + b\sin\theta)f_1(u_1, u_2) + (b\cos\theta + c\sin\theta)f_2(u_1, u_2)] \quad (5.21)$$

By denoting largest of the numbers $|a|, |b|, |c|$ by M,

our assumptions become $|f_1(u_1, u_2)| < \frac{r}{6M}$ and $|f_2(u_1, u_2)| < \frac{r}{6M}$ (5.22)

for all sufficiently small $r > 0$,

so $\frac{\partial E}{\partial u_1}F_1 + \frac{\partial E}{\partial u_2}F_2 < -r^2 + \frac{4Kr^2}{6M} = -\frac{r^2}{3} < 0$

Thus E (u_1, u_2) is a positive definite function with the property that $\frac{\partial E}{\partial u_1}F_1 + \frac{\partial E}{\partial u_2}F_2$

is negative definite.

$\therefore$ The equilibrium point is an asymptotically "**stable**".

6. Threshold Theorem

In consonance with **the principle of competitive exclusion** [Gauss (1934)], we deduce a Threshold Theorems for the lone equilibrium point.

The basic equations for the model under consideration can be re-written as

$$\frac{dN_1}{dt} = \frac{a_1N_1}{k_1}\{k_1 - N_1 - \beta_1N_2\} - h_1$$
$$\frac{dN_2}{dt} = \frac{a_2N_2}{k_2}\{k_2 - N_2 - \beta_2N_1\} - h_2 \quad (6.1)$$

where $k_1 = \frac{a_1}{\alpha_{11}}$; $k_2 = \frac{a_2}{\alpha_{22}}$; $\beta_1 = \frac{\alpha_{12}(1-k)}{a_1}$ and $\beta_2 = -\frac{\alpha_{21}(1-k)}{a_2}$.

Theorem 1. *Principle of Competitive Exclusion.*

When $\frac{k_1}{\beta_1} > k_2$ and $\frac{k_2}{\beta_2} > k_1$, then every solution of $N_1(t), N_2(t)$ of (6.1) approaches the equilibrium solution $N_1(t) = \overline{N}_1\ (\neq 0)$ and $N_2(t) = \overline{N}_2\ (\neq 0)$ as t approaches infinity. In other words, if prey and predator species are nearly identical and the microcosm can support both the members of prey and predator species depending up on the initial conditions.

Proof: The first step in our proof is to show that $N_1(t)$ and $N_2(t)$ can never become negative. To this end, observe that

$$N_1(t) = \overline{N}_1 = \frac{a_1}{2\alpha_{11}} \qquad \text{and} \qquad N_2(t) = \overline{N}_2 = \frac{a_2}{2\alpha_{22}}$$

is a solution of (6.1) for any choice of $N_1(0)$. The orbit of this solution in the $N_1 - N_2$ plane is the point (0, 0) for $N_1(0) = 0$; the line $0 < N_1 < k_1$, $N_2 = 0$ for $0 < N_1(0) < k_1$; the point $(k_1, 0)$ for $N_1(0) = k_1$; and the line $k_1 < N_1 < \infty$, $N_2 = 0$ for $N_1(0) > k_1$. Thus the N_1 axis, for $N_1 \geq 0$ is the union of four distinct orbits of (6.1). Similarly the N_2 axis, for $N_2 \geq 0$, is the union of four distinct orbits of (6.1). This implies that all solutions $N_1(t), N_2(t)$ of (6.1) which start in the first quadrant $\left(N_1(t) > 0, N_2 > 0\right)$ of the $N_1 - N_2$ plane must remain there for all future time.

The second step in our proof is to split the first quadrant into regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs. This is accomplished in the following manner.

Let l_1 and l_2 be the lines $k_1 - N_1 - \beta_1 N_2 = h_1$ and $k_2 - N_2 - \beta_2 N_1 = h_2$ respectively and the point of their intersection, is $(\overline{N}_1, \overline{N}_2)$. Observe that $\frac{dN_1}{dt}$ is negative if (N_1, N_2) lies above the line l_1 and positive if (N_1, N_2) lies below l_1. Similarly, $\frac{dN_2}{dt}$ is negative if (N_1, N_2) lies above l_2 and positive if (N_1, N_2) lies below l_2. Thus the two lines l_1 and l_2 split the first quadrant of the $N_1 - N_2$ plane into four regions in which both $\frac{dN_1}{dt}$ and $\frac{dN_2}{dt}$ have fixed signs.

$N_1(t), N_2(t)$ both increase with time in region *I*;

$N_1(t)$ increases and $N_2(t)$ decreases with time in region *II*;

$N_1(t)$ decreases and $N_2(t)$ increases with time in region *III*

and both $N_1(t)$ and $N_2(t)$ decrease with time in region *IV*. In this region both the prey predator compete with each other but do not flourish and at the same time do not get extinct.

Finally we require the following four lemmas.

Lemma 1. *Any solution of* $N_1(t), N_2(t)$ *of (*6.1*) which starts in region I at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$ *, and ultimately approach the equilibrium solution* $N_1(t) = \overline{N}_1, N_2(t) = \overline{N}_2$.(Figure 5)

Lemma 2. *Any solution of* $N_1(t), N_2(t)$ *of* (6.1) *which starts in region II at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$ *, and ultimately approach the equilibrium solution* $N_1(t) = \overline{N}_1, N_2(t) = \overline{N}_2$.(Figure 5)

Lemma 3. *Any solution of* $N_1(t), N_2(t)$ *of (6.*1*) which starts in region III at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$ *, and ultimately approach the equilibrium solution* $N_1(t) = \overline{N}_1, N_2(t) = \overline{N}_2$.(Figure 5)

Lemma 4. *Any solution of* $N_1(t), N_2(t)$ *of* (6.1) *which starts in region VI at time* $t = t_0$ *will remain in this region for all future time* $t \geq t_0$ *, and ultimately approach the equilibrium solution* $N_1(t) = \overline{N}_1, N_2(t) = \overline{N}_2$.(Figure 5)

Lemmas 1, 2, 3 and 4 state that every solution $N_1(t), N_2(t)$ of (6.1) which starts in region *I, II, III* or *VI* at time $t = t_0$ and remains there for all future time must also approach equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$ as *t* approaches infinity. Next, observe that any solution $N_1(t), N_2(t)$ of (1) which starts on l_1 or l_2 must immediately afterwards enter regions *I*, *II, III* or *VI*. Finally the solution approaches the equilibrium solution $N_1(t) = \overline{N}_1$, $N_2(t) = \overline{N}_2$. This is illustrated in Figure 6

7. Trajectories

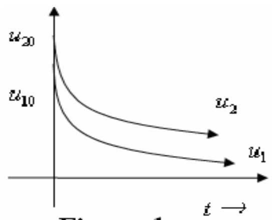

Figure 1

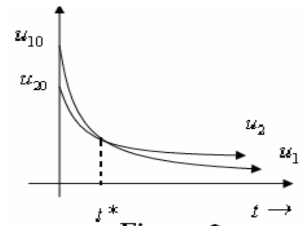

Figure 2

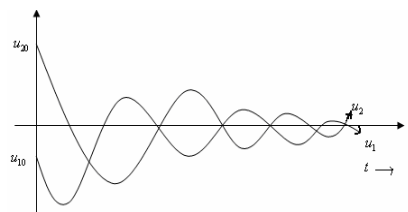

Figure 3

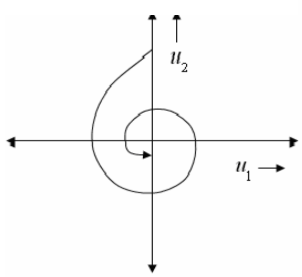

Figure 4

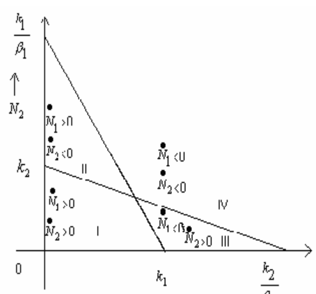

Figure 5

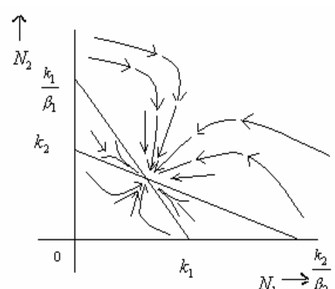

Figure 6

8. Future Work

In the present paper it is investigated that a Prey-Predator model with constant harvesting of both species, a cover for prey and a limited alternate food is supplied to the predator. There is a scope to study the model with constant harvesting of the prey species, or constant harvesting of the predator species. Further cover can be removed to the Prey and without an alternate food to the predator.

REFERENCES

[1] Michael Olinck, An introduction to Mathematical models in the social and Life Sciences , Addison Wesley, 1978.

[2] Kapur, J. N., Mathematical models in Biology and Medicine, Affiliated East-West,1985.

[3] Smith, J. M., Models in Ecology, Cambridge University Press, Cambridge, 1974.

[4] Paul Colinvaux, Ecology, John Wiley and Sons Inc., New York, 1977.

[5]. Freedman, H. I., Deterministic Mathematical Models in Population Ecology, Marcel – Decker, New York, 1980.

[6] May, R. M., Stability and complexity in Model Eco-systems, Princeton University Press, Princeton, 1973.

[7] Varma, V. S., A note on Exact solutions for a special prey-predator or competing species system, Bull. Math. Biol., **39**(1977), 619-622.

[8] Narayan, K. L., A Mathematical Study of Prey – Predator Ecological Models with a Partial Cover for the Prey and Alternative Food for the Predator, Ph.D thesis, JNTU, Hyderabad, 2005.

[9] Narayan, K.L. & Ramacharyulu,N.C.P, A Prey-Predator Model with an Alternative Food for the Predator, harvesting of both the Species and with A Gestation Period for Interaction, IJOPCM,**1(1)** (2008), 71-79.

[10] Narayan, K.L. & Ramacharyulu,N.C.P, A Prey-Predator Model with an Alternative Food for the Predator, harvesting of both the Species and. A Time Delay, Proceedings of ICM, UAE, (2008), 352-358.

[11] Narayan, K.L. & Ramacharyulu,N.C.P, A Prey-Predator Model with a cover proportional to size of the Prey and an Alternative Food for the Predator, Int. J. of Math. Sci. & Engg. Appls., **2(IV)** (2008), 129-141.

[12] Narayan, K.L. & Ramacharyulu,N.C.P, “Some Threshold Theorems for a Prey-Predator Model with Harvesting, Int.J. of Math. Sci. & Engg. Appls., **2(II)(**2008), 179-191 .

[1]Department of Mathematics & Humanities, SLC’S IET, Hyderabad-501 512, India.

[2] Department of Mathematics & Humanities, NIT, Warangal-506 004, India.

E-mail address: narayankunderu@yahoo.com