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(Short Note)

ON STRUCTURE OF H-SPACES

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Abstract. A pair (X, A) of a topological space X and a topological ring A is called an H-space, if for each closed subset F of X and $x \notin F$, there exists $f \in C_A(X)$ such that $f(x) \neq o_A$ and $F \subseteq Z(f)$ and a topological space X is called a V-space, [4], if for points a, b, c, and d of X, where $a \neq b$, there exists a continuous functions f of X into itself such that $f(a) = c$ and $f(b) = d$. In this paper we investigate some properties of H-spaces. In addition to , we show that every H-space is not a V-space.

1. Introduction

All topological spaces considered here are assumed to be Hausdroff.
If $X$is a topological space and A to be a topological ring ,then $C_A(X)$ denote the ring of all continuous function from X into A under the pointwis multiplication. if A is considered real number R with the usual topology $C_R(X)$ will simply be denoted by $C(X)$.it is shown in [2]that if X is any topological space, then there is a Tychonoff space, (that is a completely regular hausdorff space), Y such that $C(X)$ and $C(Y)$ isomorphic.

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In this paper, we investigate some structure of H-spaces.

2. **H-Space**

Definition 2.1. A pair (X, A) of a topological space X and a topological ring A is called an H-space, if for each closed subset F of X and $x \notin F$, there exists $f \in C_A(X)$ such that $f(x) \neq o_A$ and $F \subseteq Z(f)$, where o_A is the zero element of the ring A.

It is easy to see that if X is completely regular and A is path connected, or if X is 0-dimensional and A is any topological ring, then (X, A) is an H-space.

Theorem 2.1. *If $\{(X_\alpha, A_\alpha)\}_{\alpha\in I}$ is a family of H-spaces, then $(\prod_{\alpha\in I} X_\alpha, \prod_{\alpha\in I} A_\alpha)$ is also an H-space, where $\prod_{\alpha\in I} X_\alpha$ denotes the product space of the space X while $\prod_{\alpha\in I} A_\alpha$ denoted the direct product of the rings A.*

Definition 2.2. A topological space X is called a V-space, [4], if for points a, b, c, and d of X, where $a \neq b$, there exists a continuous functions f of X into itself such that $f(a) = c$ and $f(b) = d$.

K.D.Magill is shown [4], that every completely regular path connected space and every zero-dimensional space is a V-space.
As the following example, we show that if A is a topological ring such that (A, A) is a H-space, (A, A) will not be V-space.

Example 2.1. *Let X be the ring of real numbers with the usual topology, and let Y be the ring of integers with the discrete topology. Then X is path connected while Y is zero-dimensional, thus (X, X) and (Y, Y) are H-spaces. Hence $(X \times Y, X \times Y)$ is also an H-space. Since $X \times Y$ is not connected with all components homeomorphic to X, it follows from* [4, Theorem 3.5, p. 178] *that $X \times Y$ is not a V-space.*

3. **Equivalence Class on H-space**

Let X be a topological space, and A be a topological ring. For x and y in X, define $x \equiv_A y$ if and only if $f(x) = f(y)$ for each $f \in C_A(X)$. Then " $\equiv_A$ " is an equivalence relation in X. Let $[Y]_A$ be the set of all equivalence classes, and let $\varphi : X \to [Y]_A$ be the natural map. For each $f \in C_A(X)$, let $f_\varphi : [Y]_A \to R$ defined by $f_\varphi([x]) = f(x)$. Then f_φ is well-defined and $f_\varphi o \varphi = f$ for each $f \in C_A(X)$.
Let

$$[C]_A = \{f_\varphi : f \in C_A(X)\} = \{g : go\varphi \in C_A(X)\}$$

and let $\acute{\tau}$ be the weak topology on $[Y_A]$ induced by the family C_A. Note that the construction of the space $[Y]_A$ is analogous.

Theorem 3.1. *(1) The topological space* $([Y]_A, A)$ *is Hausdorff.*
(2) $([Y]_A, A)$ *is completely regular.*
(3) The mapping $\varphi : X \to ([Y]_A, \acute{\tau})$ *is continuous.*

Theorem 3.2. *If the ring A is path connected, then:*
(1) $(([Y]_A, \acute{\tau}), A)$ *is an H-space*
(2) $[Y]_A = [Y]_R$
(3) $\acute{\tau} = \tau$

Proof: Since A is assumed to be path connected while $([Y]_A, A)$ is completely regular by Theorem 2, (1) is clear.

(2) it is sufficient to show that $x \equiv_R y$ if and only if $x \neq_A y$ whenever $x, y \in X$. Let $x \equiv_R y$ and $x \neq_A y$. Then there exists $f \in C_A(X)$ such that $f(x) \neq f(y)$. Let $g \in C(A)$ such that $g(f(x)) \neq g(f(y))$ This would imply that $x \neq_R y$ since $gof \in C(X)$, a contradiction.

Conversely, if $x \equiv_A y$ but $x \not\equiv_R y$. Then there exists $f \in C(X)$ such that $f(x) \neq f(y)$. Then there exists $h \in C_A(R)$ such that $h(f(x)) = 0$ but $h(f(y)) \neq 0$. If $g = hof$, then $g \in C_A(X)$ but $g(x) \neq g(y)$ which leads to a contradiction again.

(3). Since A is completely regular $C(A)$ separates points from closed sets in A, thus sets of the form $g^{-1}(V)$, where $g \in C(A)$ and V open in R, form a subbase for the topology of A. Let $f_{\varphi}^{-1}(U)$ be a subbasic open set in $\acute{\tau}$. Then U is open in A, hence we may let $U = \bigcap_{i=1}^{n} g_i^{-1}(V_i)$ where V_i open in R. Thus we have

$$f_{\varphi}^{-1}(U) = f_{\varphi}^{-1}(\bigcap_{i=1}^{n} g_i^{-1}(V_i)) = \bigcap_{i=1}^{n} (g_i o f_i)^{-1}(V_i)$$

by (2) $\acute{\tau} \subset \tau$ (Since $[Y]_A = [Y]_R$).

Conversely, let $h^{-1}(U)$ be a sub basic open set in τ, where U is open in R and $ho\varphi \in C(X)$. Let $y \in h^{-1}(U)$. Since $([Y]_R, \tau)$ is completely regular, $(([Y]_R, \tau), A)$ is an H-space, hence there exists $f \in C([Y]_R)_A$ such that $f(y) \neq 0$ but $f([Y]_R - h^{-1}(U)) = 0$. Then $y \in f^{-1}(A - \{0\})$. Since $fo\varphi \in C_A(X)$, this shows that $h^{-1}(U) \in \acute{\tau}$. Hence $\tau \subset \acute{\tau}$

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