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COMPARISON OF RELATIVE RISK FUNCTIONS OF THE RAYLEIGH DISTRIBUTION UNDER TYPE-II CENSORED SAMPLES: BAYESIAN APPROACH *

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ABSTRACT: Based on complete as well as type II censored samples, the Bayes' estimators for the parameter and reliability function of Rayleigh distribution are obtained. These estimators are obtained on the basis of squared error loss function and LINEX loss function. Comparisons in terms of risks of those under squared error loss and LINEX loss functions with Bayes estimators relative to squared error loss function have been made. Finally, Monte Carlo simulations are performed to compare the performances of the Bayes estimates under different situations.

1. INTRODUCTION

Rayleigh distribution, which is a special case of Weibull distribution, has wide applications in lifetime data analysis especially in reliability theory and survival analysis. Polovko (1968) and Dyer and Whisenand (1973) demonstrated the importance of this distribution in electro vacuum devices and communication engineering. The origin and other aspects of this distribution can be found in Siddiqui (1962), Hirano (1986). Ariyawansa and Templeton (1984) have also discussed some of its applications. Howlader and Hossian (1995) obtained Bayes estimators and highest posterior density intervals for the scale parameter and the reliability function in case of type-II censored sampling by using Hartigan prior. Singh et al (2005) compared maximum likelihood estimators, generalized maximum likelihood estimators and Bayes estimators under type-II censored sampling of an exponentiated–Weibull distribution. They considered independent non-informative types of priors to obtain Bayes estimators. The performances of the Bayes estimators are studied by their simulated risks. Abd Elfattah *et al* (2006a) studied the efficiency of maximum likelihood estimates of the parameter of Rayleigh distribution under three cases,

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type-I, type-II and progressive type-II censored sampling schemes. Abd Elfattah *et al* (2006b) have also obtained Bayes risk under squared error loss function and maximum likelihood risk functions for complete and type II censored sampling schemes. Hendi *et al* (2007) obtained Bayes' estimators of the scale parameter, reliability function and failure rate by using non-informative prior and Hartigan prior based on upper record values.

The probability distribution function (pdf) and the reliability function of the Rayleigh distribution are respectively given by:

$$f(x \mid \sigma) = \frac{x}{\sigma^2} \exp\left(-\frac{1}{2\sigma^2} x^2\right) \quad ; x \geq 0, \sigma > 0 \tag{1}$$

and $R(t) = \overline{F(t)} = P(X > t) = \exp\left(-\frac{t^2}{2\sigma^2}\right)$; $t \geq 0$, $\sigma > 0$ (2)

In the estimation of reliability function, use of symmetric loss function may be inappropriate as recognized by Canfield (1970). Varian (1975) proposed an asymmetric loss function known as LINEX loss function which has been found to be appropriate in the situation where overestimation is more serious than underestimation or vice-versa.

The LINEX loss function for a parameter σ is given by

$$L(\Delta) = [e^{a\Delta} - a\Delta - 1], \qquad a \neq 0 \tag{3}$$

where, $\Delta = \frac{\hat{\sigma}}{\sigma} - 1$, and $\hat{\sigma}$ is an estimate of σ.

The sign and magnitude of 'a' represents the direction and degree of asymmetry respectively. The positive value of 'a' is used when over estimation is more serious than underestimation while negative value of 'a' is used in the reverse situations. If 'a' is close to zero, this loss function is approximately squared error loss and therefore almost symmetric. Several authors including Basu and Ebrahimi(1991), Rojo(1987), Soliman (2000) and Zellner(1986) have used this loss function in various estimation and prediction problems.

If we take $\Delta_1 = \hat{\sigma} - \sigma$, then $L(\Delta_1)$ is equivalent to the loss function used by Varian (1975) and Zellner (1986).Again if we take $\Delta_2 = (\frac{\hat{\sigma}}{\sigma})^2 - 1$ then $L(\Delta_2)$ is equivalent to the loss function used by Soliman (2000).

Here, we consider the natural conjugate family of priors:

$$g(\sigma) \propto \frac{\exp(-\frac{\beta}{2\sigma^2})}{\sigma^{\alpha+1}} \quad , \alpha, \beta > 0 \tag{4}$$

If β =0, α =0, we get a non-informative prior. Also, if β =0, α =2, we get the asymptotically invariant prior, proposed by Hartigan (1964).

The plan of the article is as follows: In section 2 of the present paper, we have obtained the Bayes estimators of σ taking $g(\sigma)$ as a prior distribution using the squared error loss function and LINEX loss function $L(\Delta_2)$, where $\Delta_2 = (\frac{\hat{\sigma}}{\sigma})^2 - 1$. The risk of estimators has also been obtained. Comparisons in terms of risk with the estimators of σ under squared error loss and LINEX loss functions have been made. In section 3, a numerical example has been given to compare the results. In section 4, we obtain Bayes estimator of R(t) when the LINEX loss function $L(\Delta_1)$, where $\Delta_1 = \hat{\sigma} - \sigma$, is used and compared with those corresponding to the squared error loss function and a simulation study is performed to endorse our estimation techniques in section 5.

2. BAYES ESTIMATE OF σ

A group of n components have lifetimes which follow a Rayleigh distribution. Due to the cost and time considerations, the failure times are recorded as they occur until a fixed number **r (≤ n)** of components have failed. It is quite common in life testing situations that only the first r lifetimes in a sample of n components can be obtained (Type II censoring). Let $x = (x_{1:n}, x_{2:n}, . . . , x_{r:n})$ are i.i.d. random variables, where $x_{i:n}$ is the time of the ith component to fail. Since the remaining (n-r) components have not yet failed and thus have lifetimes greater than $x_{r:n}$, the likelihood function can be written as

$$L(\mathbf{x}|\sigma) = \frac{n!}{(n-r)!}\sigma^{-2r} \exp(-\frac{T}{2\sigma^2}) \prod_{i=1}^{r} x_{i:n} \qquad ; \sigma > 0 \qquad (5)$$

Where,

$$T = \sum_{j=1}^{r-1} x_{j:n}^2 + (n-r+1)x_{r:n}^2$$

$$= \sum_{j=1}^{r} (n-j+1)(x_{j:n}^2 - x_{j-1:n}^2)$$

It is straightforward to show that $\frac{X_j^2}{2\sigma^2}$ is distributed exponential with mean 2 which implies that T is gamma with shape parameter r and scale parameter 2 or a Chi-square distribution with 2r degree of freedom giving the probability density function of T as

$$h_T(t) = \frac{1}{2^r (\sigma^2)^r \Gamma(r)} \exp(-\frac{t}{2\sigma^2}) t^{r-1} \qquad ; \quad t > 0 \qquad (6)$$

2.1. Bayes estimator of σ based on squared error loss function

Using Bayes theorem, the posterior pdf of σ is

$$\pi(\sigma | \mathbf{x}) = \frac{L(x|\sigma)\, g(\sigma)}{\int_0^\infty L(x|\sigma)\, g(\sigma)\, d\sigma}$$

$$= \frac{2(\frac{T+\beta}{2})^{\frac{2r+\alpha}{2}}}{\Gamma(\frac{2r+\alpha}{2})} \frac{\exp(-\frac{T+\beta}{2\sigma^2})}{\sigma^{2r+\alpha+1}} \;,\; \sigma\,,\; \alpha, \beta > 0 \tag{7}$$

Considering the squared error loss ($L(\hat{\sigma},\sigma) = (\hat{\sigma}-\sigma)^2$), the Bayes estimator of σ denoted by $\hat{\sigma}_{SB}$ for the above prior , given by the posterior mean of σ is

$$\hat{\sigma}_{SB} = \int_0^\infty \sigma \;\; \pi(\,\sigma \mid \mathbf{x})\, d\sigma$$

$$= \frac{\Gamma(\frac{2r+\alpha-1}{2})}{\Gamma(\frac{2r+\alpha}{2})} \;\; (\frac{T+\beta}{2})^{1/2} \tag{8}$$

2.2. Bayes estimator of σ based on LINEX Loss Function

Under the LINEX loss function (3), the posterior expectation of the loss function $L(\Delta_2)$ with respect to $\pi(\,\sigma \mid \mathbf{x})$ in (7) is

$$E[L(\Delta_2)] = \int_0^\infty \{e^{a[(\frac{\hat{\sigma}}{\sigma})^2-1]} - a[(\frac{\hat{\sigma}}{\sigma})^2 - 1] - 1\} \;\; \pi(\,\sigma \mid \mathbf{x}) d\sigma \tag{9}$$

$$= e^{-a}\, E\,[\exp\{a(\frac{\hat{\sigma}}{\sigma})^2\}] - a\, E[(\frac{\hat{\sigma}}{\sigma})^2 \text{-}1] \; - 1 \tag{10}$$

The value of $\hat{\sigma}$ that minimizes the posterior expectation of the loss function $L(\Delta_2)$, denoted by $\hat{\sigma}_{LB}$ is obtained by solving the equation:

$$\frac{\partial E[L(\Delta_2)]}{\partial \hat{\sigma}} = E\;[e^{-a}\frac{\hat{\sigma}}{\sigma^2}\exp(a(\frac{\hat{\sigma}}{\sigma})^2)] - E\;(\frac{\hat{\sigma}}{\sigma^2}) = 0 \tag{11}$$

that is, $\hat{\sigma}_{LB}$ is the solution of the equation

$$E[\frac{\hat{\sigma}_{LB}}{\sigma^2}\exp(a(\frac{\hat{\sigma}_{LB}}{\sigma})^2)] = e^a E(\frac{\hat{\sigma}_{LB}}{\sigma^2}) \tag{12}$$

provided that all expectation exists and are finite.

$$\Rightarrow \frac{\hat{\sigma}_{LB}}{\sigma^2}\int_0^\infty \exp(a(\frac{\hat{\sigma}^2_{LB}}{\sigma^2}))\frac{2(\frac{T+\beta}{2})^{\frac{2r+\alpha}{2}}}{\Gamma(\frac{2r+\alpha}{2})}\frac{\exp(-\frac{T+\beta}{2\sigma^2})}{\sigma^{2r+\alpha+1}}d\sigma$$

$$= e^a\int_0^\infty \frac{\hat{\sigma}_{LB}}{\sigma^2}\times\frac{2(\frac{T+\beta}{2})^{\frac{2r+\alpha}{2}}}{\Gamma(\frac{2r+\alpha}{2})}\frac{\exp(-\frac{T+\beta}{2\sigma^2})}{\sigma^{2r+\alpha+1}}d\sigma$$

On simplification, we get the optimal estimate of σ relative to L(Δ_2) is

$$\hat{\sigma}_{LB} = [(\frac{T+\beta}{2a})\{1-\exp(-\frac{2a}{2r+\alpha+2})\}]^{1/2} \tag{13}$$

2.3. The risk efficiency of $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under LINEX Loss L(Δ_2)

The risk functions of estimators $\hat{\sigma}_{LB}$ and $\hat{\sigma}_{SB}$ relative to L(Δ_2) are of interest. These risk functions are denoted by $R_L(\hat{\sigma}_{LB})$ and $R_L(\hat{\sigma}_{SB})$, where subscript L denotes risk relative to L(Δ_2) and are given by using $h_T(t)$ in (6) as follows:

$$R_L(\hat{\sigma}_{LB}) = E(\mathrm{L}(\Delta_2)) = \int_0^\infty \{e^{a[(\frac{\hat{\sigma}_{LB}}{\sigma})^2-1]} - a[(\frac{\hat{\sigma}_{LB}}{\sigma})^2-1]-1\}h_T(t)dT$$

$$= e^{-a} \int_0^\infty e^{a\{\frac{[(\frac{(T+\beta)}{2a})D]}{\sigma^2}\}} \frac{1}{(2\sigma^2)^r \Gamma(r)} T^{r-1} \exp(-\frac{T}{2\sigma^2}) dT$$

$$- \int_0^\infty a\{ \frac{[(\frac{(T+\beta)}{2a})D]}{\sigma^2} - 1\} \frac{1}{(2\sigma^2)^r \Gamma(r)} \exp(-\frac{T}{2\sigma^2}) T^{r-1} dT - 1$$

$$= \exp[\frac{2ar}{2r+\alpha+2} + \frac{\beta}{2\sigma^2} D - a] - (r + \frac{\beta}{2\sigma^2}) D + a - 1 \tag{14}$$

where $D = [1 - \exp(-\frac{2a}{2r+\alpha+2})]$

In the same manner, we get

$$R_L(\hat{\sigma}_{SB}) = E(L(\Delta_2)) = \int_0^\infty \{e^{a[(\frac{\hat{\sigma}_{SB}}{\sigma})^2 - 1]} - a[(\frac{\hat{\sigma}_{SB}}{\sigma})^2 - 1] - 1\} h_T(t) dT$$

$$= e^{-a} \int_0^\infty \exp[a(\frac{\frac{T}{2} A^2}{\sigma^2})] \frac{1}{(2\sigma^2)^r \Gamma(r)} T^{r-1} \exp(-\frac{T}{2\sigma^2}) dT$$

$$- \int_0^\infty [a(\frac{\frac{T}{2} A^2}{\sigma^2})] \frac{1}{(2\sigma^2)^r \Gamma(r)} T^{r-1} \exp(-\frac{T}{2\sigma^2}) dT - 1$$

$$= e^{-a} [\exp(\frac{a\beta}{2\sigma^2} A^2)(1 - aA^2)^{-r}] - aA^2 (r + \frac{\beta}{2\sigma^2}) + a - 1 \tag{15}$$

where,

$$A = \frac{\Gamma(\frac{2r+\alpha-1}{2})}{\Gamma(\frac{2r+\alpha}{2})}$$

The risk efficiency of $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under LINEX Loss L(Δ_2) may be defined as follows:

$$RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB}) = \frac{R_L(\hat{\sigma}_{SB})}{R_L(\hat{\sigma}_{LB})} \qquad (16)$$

2.4. The risk efficiency of estimators $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under squared error loss

The risk functions of the estimators $\hat{\sigma}_{LB}$ and $\hat{\sigma}_{SB}$ under squared error loss are denoted by $R_S(\hat{\sigma}_{LB})$ and $R_S(\hat{\sigma}_{SB})$ and are given by :

$$R_S(\hat{\sigma}_{LB}) = \int_0^\infty (\hat{\sigma}_{LB} - \sigma)^2 h_T(t) dT$$

$$= \int_0^\infty (\hat{\sigma}_{LB}^2 - 2\hat{\sigma}_{LB}\sigma + \sigma^2) \frac{1}{(2\sigma^2)^r \Gamma(r)} \exp(-\frac{T}{2\sigma^2}) T^{r-1} dT$$

$$= \frac{1}{2a} D \int_0^\infty (T+\beta) \frac{1}{(2\sigma^2)^r \Gamma(r)} \exp(-\frac{T}{2\sigma^2}) T^{r-1} dT$$

$$- 2\sigma \frac{1}{2a} D \int_0^\infty (T+\beta)^{1/2} \frac{1}{(2\sigma^2)^r \Gamma(r)} \exp(-\frac{T}{2\sigma^2}) T^{r-1} dT + \sigma^2$$

On simplification, we get,

$$R_S(\hat{\sigma}_{LB}) \cong \phi_1 \beta + 2r\sigma^2 \phi_1 - 2\sigma(\phi_1)^{1/2} \frac{\exp(\frac{\beta}{2\sigma^2})}{\Gamma(r)} \times$$

$$[\frac{\zeta_1}{2\sigma^2} - \frac{(r-1)\beta}{(2\sigma^2)^{3/2}} \zeta_2 + \frac{(r-1)(r-2)}{2!(2\sigma^2)^{5/2}} \beta^2 \zeta_3 - \ldots] + \sigma^2 \qquad (17)$$

where,

$$\varphi_1 = \frac{1}{2a}[1-\exp(-\frac{2a}{2r+\alpha+2})]$$

$$\zeta_1 = \Gamma\{(r+\frac{1}{2}), \frac{\beta}{2\sigma^2}\}$$

$$\zeta_2 = \Gamma\{(r-\frac{1}{2}), \frac{\beta}{2\sigma^2}\}$$

$$\zeta_3 = \Gamma\{(r-\frac{3}{2}), \frac{\beta}{2\sigma^2}\}$$

Again,

$$R_S(\hat{\sigma}_{SB}) = \int_0^\infty (\hat{\sigma}_{SB}-\sigma)^2 h_T(t)dT$$

$$= \int_0^\infty (\hat{\sigma}_{SB}^2 - 2\hat{\sigma}_{SB}\,\sigma + \sigma^2)\frac{1}{(2\sigma^2)^r\Gamma(r)} T^{r-1} \exp(-\frac{T}{2\sigma^2})\, dT$$

$$= \int_0^\infty [\{(\frac{T+\beta}{2})^{1/2} A\}^2 - 2\{(\frac{T+\beta}{2})^{1/2} A\}\sigma + \sigma^2] \frac{1}{(2\sigma^2)^r\Gamma(r)} \exp(-\frac{T}{2\sigma^2}) T^{r-1} dT$$

On simplification, we get,

$$R_S(\hat{\sigma}_{SB}) = \sigma^2 + \frac{A^2}{2}(2r\sigma^2+\beta) - \sqrt{2}A\sigma\frac{\exp(\frac{\beta}{2\sigma^2})}{\Gamma(r)}$$

$$\times[\frac{\zeta_1}{2\sigma^2} - \frac{(r-1)\beta}{(2\sigma^2)^{3/2}}\zeta_2 + \frac{(r-1)(r-2)}{2!(2\sigma^2)^{5/2}}\beta^2\zeta_3 - \ldots] \quad (18)$$

The efficiency of $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under squared error loss is defined as:

$$RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB}) = \frac{R_S(\hat{\sigma}_{SB})}{R_S(\hat{\sigma}_{LB})} \quad (19)$$

3. NUMERICAL EXAMPLE

To compare the proposed estimator $\hat{\sigma}_{LB}$ with the estimator $\hat{\sigma}_{SB}$, the risk functions are computed so as to see whether $\hat{\sigma}_{LB}$ out performs $\hat{\sigma}_{SB}$ under LINEX loss L(Δ_2) and how $\hat{\sigma}_{LB}$ performs as compared to $\hat{\sigma}_{SB}$ when true loss is squared error. A comparison of this type may be needed to check whether an estimator is inadmissible under some loss function. If it is so, the estimator would not be used for the losses specified by that loss function. For this purpose the risk efficiency have been computed.

A random sample of size n = 20 were generated from (1) with σ = 1, and ordered as follows:

0.2526, 0.4183, 0.4864, 0.5920, 0.6947, 0.7246, 0.7723, 0.8271, 1.0139, 1.1529, 1.1715, 1.2008, 1.2313, 1.5881, 1.9079, 1.9221, 1.9659, 2.0024, 2.0878, 3.3424

Three sets of values: (20, 20), (20, 15), (20,10) were taken for (n, r), to provide complete sample, 25% and 50% censored observations, respectively. We compute and report the estimates and their corresponding risk efficiencies for σ. Tables 1 – 4 show Bayes LINEX estimator loss $\hat{\sigma}_{LB}$, Bayes squared error estimator loss $\hat{\sigma}_{SB}$, risk efficiency $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ of $\hat{\sigma}_{LB}$ with respect to $\hat{\sigma}_{SB}$ under LINEX loss and risk efficiency $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under squared error loss.

From tables 1-4, we note:

i) The risk efficiency $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ is greater than one for all values of a, (a = ± 0.5, ± 1, ± 2), this conclusion is valid for both complete and censored samples, which indicates that the proposed estimator $\hat{\sigma}_{LB}$ is preferable to $\hat{\sigma}_{SB}$ (see Table-1).

ii) The risk efficiency $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ is greater than one for a ≥ 0.5 (see table 2). Thus the proposed estimator $\hat{\sigma}_{LB}$ performs better than the Bayes estimator $\hat{\sigma}_{SB}$ with proper choice of a.

iii) For Hartigan prior, we see that (except for a = 2), under squared error loss, the proposed estimator $\hat{\sigma}_{LB}$ has smaller risk than $\hat{\sigma}_{SB}$ (see Table-2).

iv) The values of the risk efficiencies RE_L and RE_S are very sensitive for variation in 'a'.

v) In general, the values of the risk efficiencies RE_L are very sensitive for variation in α and β .

TABLE 1: The estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for α = β =0

a	Complete sample			25% Censoring			50% Censoring		
	$\hat{\sigma}_{SB}$ =1.0575			$\hat{\sigma}_{SB}$=1.0947			$\hat{\sigma}_{SB}$=1.0068		
	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S
2	0.9889	1.4745	1.0684	1.0018	1.6889	1.0910	0.8830	2.2411	1.1362
1	1.0006	1.2906	1.0564	1.0173	1.4052	1.0753	0.9029	1.6735	1.1133
0.5	1.0065	1.2203	1.0504	1.0252	1.3117	1.0673	0.9131	1.4911	1.1016
-0.5	1.0186	1.1167	1.0382	1.0413	1.1646	1.0513	0.9340	1.2522	1.0780
-1	1.0247	1.0826	1.0321	1.0496	1.1097	1.0433	0.9448	1.1709	1.0661
-2	1.0371	1.0305	1.0198	1.0644	1.0399	1.0270	0.9671	1.0631	1.0419

TABLE 2: The estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for α = 2, β =0

a	Complete sample			25% Censoring			50% Censoring		
	$\hat{\sigma}_{SB}$ =1.0311			$\hat{\sigma}_{SB}$=1.0583			$\hat{\sigma}_{SB}$ =0.9565		
	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S
2	0.9672	1.1242	1.0634	0.9736	1.1761	1.0825	0.8485	1.2940	1.1175
1	0.9781	1.0369	1.0524	0.9878	1.0561	1.0682	0.8660	1.0917	1.0977
0.5	0.9837	1.0164	1.0468	0.9950	1.0123	1.0610	0.8750	1.0336	1.0876
-0.5	0.9949	0.9683	1.0355	1.0098	0.9639	1.0465	0.8934	0.9516	1.0672
-1	1.0006	0.9643	1.0298	1.0173	0.9496	1.0391	0.9029	0.9345	1.0569
-2	1.0122	0.9591	1.0184	1.0326	0.9478	1.0244	0.9223	0.9283	1.0359

TABLE 3: The estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for α = 1, β =1

a	Complete sample			25% Censoring			50% Censoring		
	$\hat{\sigma}_{SB}$ =1.0561			$\hat{\sigma}_{SB}$=1.0917			$\hat{\sigma}_{SB}$=1.0065		
	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S
2	0.9892	1.4541	1.0667	1.0017	1.6469	1.0881	0.8880	2.1163	1.1298
1	1.0006	1.2768	1.0550	1.0168	1.3854	1.0728	0.9071	1.6182	1.1078
0.5	1.0064	1.2143	1.0491	1.0245	1.2877	1.0651	0.9169	1.4563	1.0967
-0.5	1.0182	1.1228	1.0373	1.0401	1.1622	1.0496	0.9371	1.2381	1.0742
-1	1.0241	1.0826	1.0313	1.0481	1.1070	1.0418	0.9474	1.1643	1.0628
-2	1.0363	1.0299	1.0194	1.0643	1.0402	1.0261	0.9687	1.0623	1.0398

TABLE 4: The estimators $\hat{\sigma}_{LB}$, $\hat{\sigma}_{SB}$, the risk efficiencies $RE_L(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ and $RE_S(\hat{\sigma}_{LB}, \hat{\sigma}_{SB})$ under the prior g(σ) for α =0, β =2

a	Complete sample			25% Censoring			50% Censoring		
	$\hat{\sigma}_{SB}$ =1.0818			$\hat{\sigma}_{SB}$=1.1263			$\hat{\sigma}_{SB}$ = 1.0591		
	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S	$\hat{\sigma}_{LB}$	RE_L	RE_S
2	1.0116	1.8783	1.0701	1.0307	2.3040	1.0941	0.9289	3.4698	1.1433
1	1.0236	1.5829	1.0578	1.0466	1.8293	1.0778	0.9498	2.4314	1.1191
0.5	1.0297	1.4643	1.0516	1.0548	1.6667	1.0696	0.9605	2.0980	1.1068
-0.5	1.0420	1.2982	1.0392	1.0714	1.4054	1.0530	0.9826	1.6381	1.0819
-1	1.0482	1.2217	1.0329	1.0799	1.3043	1.0446	0.9939	1.4812	1.0693
-2	1.0609	1.1110	1.0203	1.0971	1.1538	1.0278	1.0173	1.2433	1.0439

4. BAYES ESTIMATOR OF $\overline{F(t)}$

Let $\gamma = \overline{F(t)}$ be the probability that a system will survive a specified mission time t.

By substituting $\sigma^2 = \dfrac{t^2}{-2\log\gamma}$ in (7), we obtain the posterior p.d.f. of γ as :

$$\pi_1(\gamma|x) = \left(\frac{(\frac{T+\beta}{2})^{\frac{2r+\alpha}{2}}}{\Gamma(\frac{2r+\alpha}{2})} \right) \frac{\exp\left(-\frac{T+\beta}{(-\frac{t^2}{\log\gamma})} \right)}{\left((-\frac{t^2}{2\log\gamma})^{\frac{2r+\alpha+2}{2}} \right)} (\frac{t^2}{2}) (\log\gamma)^{-2} \frac{1}{\gamma}$$

$$= \frac{1}{\Gamma(\frac{2r+\alpha}{2})} (\frac{T+\beta}{t^2})^{\frac{2r+\alpha}{2}} \gamma^{\frac{2r+\alpha}{2}-1} (-\log\gamma)^{\frac{2r+\alpha}{2}-1}, \quad 0<\gamma<1 \qquad (20)$$

Using the convex loss function $L(\Delta_1)$, $\Delta_1 = \hat{\gamma} - \gamma$, it is seen that this loss function is quite asymmetric when a =1 with overestimation being more serious than underestimation. Also, when $a < 0$, $L(\Delta_1)$ rises almost exponentially and almost linearly when $\Delta_1 < 0$ and almost linearly when $\Delta_1 > 0$. For small values of $|a|$ the loss function $L(\Delta_1)$ is approximately squared error loss and therefore almost symmetric. For more details about $L(\Delta_1)$ see Zellner (1986).

The posterior expectation of the LINEX loss function $L(\Delta_1)$ is:

$$E_\gamma \left[L(\Delta_1) \right] = \int_0^1 [e^{a(\hat{\gamma}-\gamma)} - a((\hat{\gamma}-\gamma) \;\; - 1] \; \pi_1(\gamma|\text{x}) \, d\gamma$$

$$= e^{a\hat{\gamma}} E_\gamma(e^{-a\gamma}) + E_\gamma(a\gamma) - a\hat{\gamma} - 1$$

For a minimum to exist at $\Delta_1 = 0$

$$\frac{\partial E_\gamma [L(\Delta_1)]}{\partial \hat{\gamma}} = a\, e^{a\hat{\gamma}} E_\gamma(e^{-a\gamma}) - a = 0$$

$$\Rightarrow E_\gamma(e^{-a\gamma}) = e^{-a\hat{\gamma}}$$

The Bayes estimator of γ relative to $L(\Delta_1)$, denoted by $\hat{\gamma}_{LB}$ and is given by (after simple algebra)

$$\hat{\gamma}_{LB} = -\frac{1}{a}\log\left[\sum_{j=0}^{\alpha}\frac{(-a)^j}{j!}\left(1+\frac{jt^2}{T+\beta}\right)^{-\frac{2r+\alpha}{2}}\right] \tag{21}$$

Under the squared error loss function ($L(\hat{\gamma},\gamma) = (\hat{\gamma}-\gamma)^2$), the Bayes' estimator of γ, denoted by $\hat{\gamma}_{SB}$ is given by

$$\hat{\gamma}_{SB} = \int_0^1 \gamma\, \pi_1(\gamma \mid x)\, d\gamma$$

$$= \int_0^1 \gamma \frac{1}{\Gamma(\frac{2r+\alpha}{2})}\left(\frac{T+\beta}{t^2}\right)^{\frac{2r+\alpha}{2}} \gamma^{\frac{2r+\alpha}{2}-1}(-\log\gamma)^{\frac{2r+\alpha}{2}-1}\, d\gamma$$

After simple algebra, the Bayes' estimator of γ is obtained as:

$$\hat{\gamma}_{SB} = \left(1+\frac{t^2}{T+\beta}\right)^{-\frac{2r+\alpha}{2}} \tag{22}$$

It can be seen that the risk functions relative to loss function $L(\Delta_1)$ do not exist. Let us consider the set of data generated in example 3, let t = 1, we compute $\hat{\gamma}_{LB}$ and $\hat{\gamma}_{SB}$, the results with the corresponding values of '*a*' are given in table-5.

It can be seen from table-5 that, the Bayes estimates of reliability function relative to $L(\Delta_1)$ for positive value of '*a*' are lower than the Bayes estimates under squared error loss function, this conclusion is valid for both complete and censored samples.

TABLE 5: Bayes estimation of reliability function based on LINEX loss ($\hat{\gamma}_{LB}$) and squared error loss ($\hat{\gamma}_{SB}$) for the variations in 'a', 'α' and 'β'(t=1, the true value $\gamma_{t=1}$=0.607)

a	Complete Sample				25% Censoring				50% Censoring			
	$\hat{\gamma}_{SB}$ = 0.632	$\hat{\gamma}_{SB}$ = 0.618	$\hat{\gamma}_{SB}$ = 0.631	$\hat{\gamma}_{SB}$ = 0.645	$\hat{\gamma}_{SB}$ = 0.649	$\hat{\gamma}_{SB}$ = 0.630	$\hat{\gamma}_{SB}$ = 0.648	$\hat{\gamma}_{SB}$ = 0.664	$\hat{\gamma}_{SB}$ = 0.595	$\hat{\gamma}_{SB}$ = 0.565	$\hat{\gamma}_{SB}$ = 0.596	$\hat{\gamma}_{SB}$ = 0.625
	$\hat{\gamma}_{LB}$ α=0, β =0	$\hat{\gamma}_{LB}$ α=2, β =0	$\hat{\gamma}_{LB}$ α=1, β =1	$\hat{\gamma}_{LB}$ α=0, β =2	$\hat{\gamma}_{LB}$ α=0, β=0	$\hat{\gamma}_{LB}$ α=2, β =0	$\hat{\gamma}_{LB}$ α=1, β =1	$\hat{\gamma}_{LB}$ α=0, β =2	$\hat{\gamma}_{LB}$ α=0, β =0	$\hat{\gamma}_{LB}$ α=2, β =0	$\hat{\gamma}_{LB}$ α=1, β =1	$\hat{\gamma}_{LB}$ α=0 , β=2
2	.628	.613	.627	.641	.644	.625	.642	.659	.586	.556	.587	.616
1	.630	.615	.629	.643	.646	.628	.645	.662	.590	.560	.591	.621
0.5	.631	.616	.630	.644	.647	.629	.646	.663	.593	.563	.593	.623
-.5	.633	.619	.632	.646	.650	.632	.649	.665	.597	.567	.598	.627
-1	.634	.620	.633	.647	.651	.633	.650	.667	.600	.569	.600	.629
-2	.636	.622	.635	.649	.654	.635	.652	.669	.604	.574	.604	.633

5. SIMULATION RESULTS

One sample does not tell us much. We generated N = 1000 sample of sizes n =10, 20, 30 from (1) with σ =1.

Define the Monte Carlo estimator of a parameter θ by

$$\hat{\sigma} = \sum_{i=1}^{N} \hat{\sigma}_i \,/\, N$$

and the root- mean-square -error

$$RMSE(\hat{\sigma}) = \sqrt{\sum_{i=1}^{N} (\hat{\sigma}_i - \sigma_o)^2 / N}\ , \quad \sigma_o = \text{true value of } \sigma$$

Compute $\hat{\gamma}_{LB}$ and $\hat{\gamma}_{SB}$ at t = 1 for the variations in 'a', 'α' and 'β'. The results are in Table-6.The entries within parentheses represents the corresponding RMSE. From Table-6, we see that the Bayes estimators of reliability function based on LINEX loss have smaller RMSE than that of squared error loss when a < 0.5 irrespective of sample sizes. Therefore, we conclude that in situations involving reliability estimation, asymmetric loss function is more appropriate than squared error loss function when a<0.5.

TABLE 6: Simulation estimates of the mean value of Bayes estimators of reliability function based on LINEX loss ($\hat{\gamma}_{LB}$) and squared error loss ($\hat{\gamma}_{SB}$) for the variations in 'a', 'α' and 'β' (t=1, the true value $\gamma_{t=1}$=0.607)

n	a	α = 0, β =0	α = 2, β =0	α =1, β =1	α = 0, β =2
		$\hat{\gamma}_{SB}$ = 0.5956 (0.1020)	$\hat{\gamma}_{SB}$ = 0.5682 (0.1086)	$\hat{\gamma}_{SB}$ = 0.5820 (0.1033)	$\hat{\gamma}_{SB}$ = 0.6267 (0.0846)
10		$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$
	2	0.5868 (0.1054)	0.5595 (0.1136)	0.5732 (0.1077)	0.6185 (0.0852)
	1	0.5866 (0.1044)	0.5585 (0.1133)	0.5823 (0.0981)	0.6169 (0.0837)
	.5	0.5834 (0.0990)	0.5576 (0.1089)	0.5840 (0.0955)	0.6153 (0.0808)
	-.5	0.5868 (0.0967)	0.5639 (0.1075)	0.5933 (0.0900)	0.6229 (0.0835)
	-1	0.5941 (0.0928)	0.5659 (0.1072)	0.5983 (0.0886)	0.6263 (0.0822)
	-2	0.5963 (0.0925)	0.5677 (0.1044)	0.6021 (0.0861)	0.6269 (0.0816)
20	a	$\hat{\gamma}_{SB}$ =0.6070 (0.0693)	$\hat{\gamma}_{SB}$ = 0.5868 (0.0721)	$\hat{\gamma}_{SB}$ = 0.6051 (0.0657)	$\hat{\gamma}_{SB}$ = 0.6202 (0.0649)
		$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$
	2	0.6027 (0.0703)	0.5822 (0.0704)	0.6007 (0.0668)	0.6159 (0.0650)
	1	0.6016 (0.0680)	0.5813 (0.0732)	0.6026 (0.0657)	0.6130 (0.0642)
	.5	0.6010 (0.0676)	0.5805 (0.0694)	0.6029 (0.0643)	0.6117 (0.0632)
	-.5	0.6053 (0.0652)	0.5887 (0.0686)	0.6070 (0.0634)	0.6149 (0.0639)
	-1	0.6070 (0.0618)	0.5894 (0.0667)	0.6073 (0.0622)	0.6173 (0.0639)
	-2	0.6068 (0.0617)	0.5900 (0.0664)	0.6084 (0.0605)	0.6184 (0.0641)
30	a	$\hat{\gamma}_{SB}$ =0.6150 (0.0538)	$\hat{\gamma}_{SB}$ = 0.5997 (0.0552)	$\hat{\gamma}_{SB}$ = 0.6006 (0.0547)	$\hat{\gamma}_{SB}$ = 0.6158 (0.0519)
		$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$	$\hat{\gamma}_{LB}$
	2	0.6121 (0.0539)	0.5968 (0.0562)	0.5976 (0.0556)	0.6129 (0.0520)
	1	0.6078 (0.0537)	0.5921 (0.0554)	0.6028 (0.0550)	0.6103 (0.0502)
	.5	0.6010 (0.0530)	0.5917 (0.0549)	0.6035 (0.0532)	0.6102 (0.0497)
	-.5	0.6023 (0.0530)	0.5939 (0.0520)	0.6064 (0.0529)	0.6172 (0.0492)
	-1	0.6096 (0.0517)	0.5986 (0.0512)	0.6100 (0.0526)	0.6180 (0.0482)
	-2	0.6102 (0.0516)	0.6053 (0.0499)	0.6111 (0.0511)	0.6195 (0.0467)

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