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ON SOME NEW INEQUALITIES RELATED TO FEJÉR'S INEQUALITY FOR SUPERQUADRATIC FUNCTIONS

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Abstract. In this paper some new inequalities related to Fejér's inequality and Hermite-Hadamard's inequality for superquadratic functions are established. We also get refinements of some known results established in [20] when superquadratic function is non-negative and hence convex.

1. Introduction

Let $f : I \longrightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, $a, b \in I$ with $a < b$ be a convex function then the following integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is famous in literature and is known as Hermite-Hadamard inequality. This inequality has various applications for generalized means, information measures and quadrature rules. The weighted generalization of (1.1) is the following inequality:

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx \leq \int_a^b f(x)p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx, \tag{1.2}$$

where f as defined above, $p : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric about $x = \frac{a+b}{2}$. The inequality (1.2) is known as Fejér's inequality.

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Let us now consider the following mappings on $[0,1]$ and quote some results established by Tseng et al. and S. S. Dragomir from [6, 9, 17, 18, 19, 20]:

$$G(t)=\frac{1}{2}\left[f\left(ta+(1-t)\frac{a+b}{2}\right)+f\left(tb+(1-t)\frac{a+b}{2}\right)\right],$$

$$H(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

$$H_p(t)=\frac{1}{b-a}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)p(x)dx,$$

$$I(t)=\int_a^b \frac{1}{2}\left[f\left(t\frac{x+a}{2}+(1-t)\frac{a+b}{2}\right)\right.$$
$$\left.+f\left(t\frac{x+b}{2}+(1-t)\frac{a+b}{2}\right)\right]p(x)dx,$$

$$F(t)=\int_a^b\int_a^b f\left(tx+(1-t)y\right)dxdy,$$

$$K(t)=\frac{1}{4}\int_a^b\int_a^b\left[f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)+f\left(t\frac{x+a}{2}+(1-t)\frac{y+b}{2}\right)\right.$$
$$\left.+f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{y+b}{2}\right)\right]p(x)p(y)dxdy,$$

$$L(t)=\frac{1}{2(b-a)}\int_a^b\left[f\left(ta+(1-t)x\right)+f\left(tb+(1-t)x\right)\right]dx,$$

$$L_p(t)=\frac{1}{2}\int_a^b\left[f\left(ta+(1-t)x\right)+f\left(tb+(1-t)x\right)\right]p(x)dx,$$

$$S_p(t)=\frac{1}{4}\int_a^b\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(ta+(1-t)\frac{x+b}{2}\right)\right.$$
$$\left.+f\left(tb+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]p(x)dx$$

and

$$N(t)=\frac{1}{2}\int_a^b\left[f\left(ta+(1-t)\frac{x+a}{2}\right)+f\left(tb+(1-t)\frac{x+b}{2}\right)\right]p(x)dx$$

where $f : [a, b] \to \mathbb{R}$ is a convex function and $p : [a, b] \to \mathbb{R}$ is non-negative integrable and symmetric about $x = \frac{a+b}{2}$.

S. S. Dragomir [6] established the following Hermite-Hadamard-type inequalities for the functions H and F:

Theorem 1.1. [6] *Let f, H be defined as above. Then H is convex, increasing on $[0, 1]$ and for all $t \in [0, 1]$, we have*

$$f\left(\frac{a+b}{2}\right) = H(0) \le H(t) \le H(1) = \frac{1}{b-a}\int_a^b f(x)dx. \tag{1.3}$$

Theorem 1.2. [6] *Let f, F be defined as above. Then*

(1) *F is convex on $[0, 1]$, symmetric about $\frac{1}{2}$, F is decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$, and for all $t \in [0, 1]$,*

$$\sup_{t\in[0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\inf_{t\in[0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2}\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)dxdy.$$

(2) *We have:*

$$f\left(\frac{a+b}{2}\right) \le F\left(\frac{1}{2}\right) \quad ; \qquad H(t) \le F(t), \quad t \in [0, 1]\,. \tag{1.4}$$

As a weighted generalization of Theorem 1, Tseng et al. [17] gave the following Fejér type inequalities related to the functions I and N:

Theorem 1.3. [17] *Let f, p, I, N be defined as above. Then I, N are convex, increasing on* $[0,1]$ *and for all* $t \in [0,1]$, *we have*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)dx = I(0) \le I(t) \le I(1) \tag{1.5}$$
$$= \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx$$
$$= N(0) \le N(t) \le N(1) = \frac{f(a)+f(b)}{2}\int_a^b p(x)dx.$$

Dragomir et al. [9] established the following Hermite-Hadamard-type inequalities related to the functions H, G, L:

Theorem 1.4. *Let f, H, G, L be defined as above. Then G is convex, increasing on* $[0,1]$, *L is convex on* $[0,1]$ *and for all* $t \in [0,1]$, *we have:*

$$H(t) \le G(t) \le L(t) \le \frac{1-t}{b-a}\int_a^b f(x)dx + t\cdot\frac{f(a)+f(b)}{2} \le \frac{f(a)+f(b)}{2}. \tag{1.6}$$

Tseng et al. [18, 19] established the following theorems related to Fejér-type inequalities for the functions G, H_p, L_p, I, S_p which provide a weighted generalization of the above theorem:

Theorem 1.5. [18] *Let f , p, G, H_p, L_p be defined as above. Then L_p is convex, increasing on* $[0,1]$, *and for all* $t \in [0,1]$, *we have*

$$H_p(t) \le G(t)\int_a^b p(x)dx \le L_p(t) \le (1-t)\int_a^b f(x)p(x)dx \tag{1.7}$$
$$+ t.\frac{f(a)+f(b)}{2}\int_a^b p(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b p(x)dx.$$

Theorem 1.6. [19] *Let f , p, G, I, S_p be defined as above. Then S_p is convex, increasing on $[0,1]$, and for all $t \in [0,1]$, we have*

(1.8)
$$I(t) \leq G(t)\int_a^b p(x)dx \leq S_p(t) \leq (1-t)\int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx + t\cdot\frac{f(a)+f(b)}{2}\int_a^b p(x)dx \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)dx.$$

Related to the functions H, F, L Dragomir proved the following Hermite-Hadamard-type inequalities in [7]:

Theorem 1.7. [7] *Let H, F, L be defined as above. Then we have the inequalities*

$$0 \leq F(t) - H(t) \leq L(1-t) - F(t), \tag{1.9}$$

for all $t \in [0,1]$.

Finally, Tseng et al. in [20] established the following Fejér and Hermite-Hadamard-type inequalities for the functions H, H_p, I, F, K, L_p, S_p which also provide weighted generalization of Theorem 2 and Theorem 7.

Theorem 1.8. [20] *Let f , p, I and K be defined as above. Then:*

(1) *K is convex on $[0,1]$ and symmetric about $\frac{1}{2}$.*
(2) *K is decreasing on $\left[0,\frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2},1\right]$,*

$$\sup_{t\in[0,1]} K(t) = K(0) = K(1) = \int_a^b \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx\int_a^b p(x)dx$$

and

$$\inf_{t\in[0,1]} K(t) = K\left(\frac{1}{2}\right) = \int_a^b\int_a^b \frac{1}{4}\left[f\left(\frac{x+y+2a}{4}\right)+2f\left(\frac{x+y+a+b}{4}\right) f\left(\frac{x+y+2b}{4}\right)\right]p(x)p(y)dxdy.$$

(3) *We have*

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2 \le K\left(\frac{1}{2}\right) \tag{1.10}$$

and

$$I(t)\int_a^b p(x)dx \le K(t), \tag{1.11}$$

for all $t \in [0,1]$.

Theorem 1.9. [20] *Let* f, p, K, I, S_p *be defined as above. Then we have the following inequality:*

$$0 \le K(t) - I(t)\int_a^b p(x)dx \le S_p(t)\int_a^b p(x)dx - K(t), \tag{1.12}$$

for all $t \in [0,1]$.

Theorem 1.10. [20] *Let* f, p, G, I, K, S_p, L_p *be defined as above. Then the following inequalities hold for all* $t \in [0,1]$*:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2 &\le I(t)\int_a^b p(x)\,dx \le K(t) \\ &\le \frac{1}{2}\left[I(t)+S_p(1-t)\right]\int_a^b p(x)\,dx \le \frac{1}{2}\left[G(t)\int_a^b p(x)\,dx + S_p(1-t)\right]\int_a^b p(x)\,dx \\ &\le \frac{1}{2}\left[L_p(t)+S_p(1-t)\right]\int_a^b p(x)\,dx \le \frac{1}{2}\Bigg[(1-t)\int_a^b f(x)p(x)\,dx \\ &+t\int_a^b \frac{1}{2}\left\{f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right\}p(x)dx + \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx\Bigg]\int_a^b p(x)\,dx \\ &\le \frac{f(a)+f(b)}{2}\left(\int_a^b p(x)\,dx\right)^2 \end{aligned} \tag{1.13}$$

and

$$
\begin{aligned}
(1.14)\quad & f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2 \le I(t)\int_a^b p(x)\,dx \le K(t) \\
& \le \frac{1}{2}\left[I(t)+S_p(1-t)\right]\int_a^b p(x)\,dx \le \frac{1}{2}\left[G(t)\int_a^b p(x)\,dx + S_p(1-t)\right]\int_a^b p(x)\,dx \\
& \le \frac{1}{2}\left[S_p(t)+S_p(1-t)\right]\int_a^b p(x)\,dx \le \frac{1}{2}\left[\int_a^b \frac{1}{2}\left\{f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right\}p(x)dx \right.\\
& \left. +\frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx\right]\int_a^b p(x)\,dx \le \frac{f(a)+f(b)}{2}\left(\int_a^b p(x)\,dx\right)^2.
\end{aligned}
$$

Tseng et al. used the following lemma to prove their results:

Lemma 1.1. [18, p. 3] $f:[a,b]\longrightarrow\mathbb{R}$ *be convex function and let* $a\le A\le C\le D\le B\le b$ *with* $A+B=C+D$*. Then*

$$f(A)+f(B)\le f(C)+f(D).$$

The main purpose of the present paper is to establish new Fejér and Hermite-Hadamard type inequalities for superquadratic function. We also get refinements of Theorem 8, Theorem 9 and Theorem 10 and hence as a consequence we also get refinements of Theorem 7 and Theorem 2 when superquadratic function is non-negative and hence convex.

2. Main Results

In this section we prove our main results by using the same techniques as used in [20] and [2]. We also assume that all the considered integrals in this section exist. Let us first recall the definition, some of properties and results related to superquadratic functions to be used in the sequel.

Definition 2.1. [3, Defintion 2.1] Let $I=[0,a]$ or $[0,\infty)$ be an interval in $\mathbb{R}$. A function $f:I\longrightarrow\mathbb{R}$ is superquadratic if for each x in I there exists a real number

$C(x)$ such that

$$f(y) - f(x) \geq C(x)(y-x) + f(|y-x|) \tag{2.1}$$

for all $y \in I$. If $-f$ is superquadratic then f is called subquadratic.

For examples of superquadratic functions see [2, p. 1049].

Theorem 2.1. [3, Theorem 2.3] *The inequality*

$$f\left(\int g d\mu\right) \leq \int \left(f(g(s)) - f\left(\left|g(s) - \int g d\mu\right|\right)\right) d\mu(s) \tag{2.2}$$

holds for all probability measure μ and all non-negative μ-integrable function g, if and only if f is superquadratic.

The following discrete version of the above theorem will be helpful in the sequel of the paper:

Lemma 2.1. [2, Lemma A, p.1049] *Suppose that f is superquadratic. Let $x_r \geq 0$, $1 \leq r \leq n$, and let $\bar{x} = \sum_{r=1}^{n} \lambda_r x_r$ where $\lambda_r \geq 0$ and $\sum_{r=1}^{n} \lambda_r = 1$. Then*

$$\sum_{r=1}^{n} \lambda_r f(x_r) \geq f(\bar{x}) + \sum_{r=1}^{n} \lambda_r f(|x_r - \bar{x}|). \tag{2.3}$$

The following Lemma shows that positive superquadratic functions are also convex:

Lemma 2.2. [3, Lemma 2.2] *Let f be superquadratic function with $C(x)$ as in Definition 1. Then*

(1) $f(0) \leq 0$.
(2) *If $f(0) = f^{'}(0) = 0$ then $C(x) = f'(x)$ whenever f is differentiable at $x > 0$.*
(3) *If $f \geq 0$, then f convex and $f(0) = f'(0) = 0$.*

In [4] a converse of Jensen's inequality for superquadratic functions was proved:

Theorem 2.2. [4, Theorem 1] *Let* (Ω, A, μ) *be a measurable space with* $0 < \mu(\Omega) < \infty$ *and let* $f : [0, \infty) \to \mathbb{R}$ *be a superquadratic function. If* $g : \Omega \to [m, M] \subseteq [0, \infty)$ *is such that* g*,* $f \circ g \in L_1(\mu)$*, then we have for* $\bar{g} = \frac{1}{\mu(\Omega)} \int g d\mu$*,*

$$\begin{aligned}(2.4)\quad \frac{1}{\mu(\Omega)} \int f(g) d\mu \leq & \frac{M - \bar{g}}{M - m} f(m) + \frac{\bar{g} - m}{M - m} f(M) \\ & - \frac{1}{\mu(\Omega)} \frac{1}{M - m} \int ((M - g) f(g - m) + (g - m) f(M - g)) d\mu.\end{aligned}$$

The discrete version of this theorem is:

Theorem 2.3. [4, Theorem 2] *Let* $f : [0, \infty) \to \mathbb{R}$ *be a superquadratic function. Let* $(x_1, ..., x_n)$ *be an* n*-tuple in* $[m, M]^n$ $(0 \leq m \leq M < \infty)$*, and* $(p_1, ..., p_n)$ *be a non-negative* n*-tuple such that* $P_n = \sum_{i=1}^{n} p_i > 0$*. Denote* $\bar{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i$*, then*

$$\begin{aligned}(2.5)\quad \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \leq & \frac{M - \bar{x}}{M - m} f(m) + \frac{\bar{x} - m}{M - m} f(M) \\ & - \frac{1}{P_n (M - m)} \sum_{i=1}^{n} p_i [(M - x_i) f(x_i - m) + (x_i - m) f(M - x_i)]\end{aligned}$$

For recent results on Fejér and Hermite-Hadamard type inequalities for superquadratic functions, we refer interested readers to [5], [4] and [2].

In order to prove our main results we go through some calculations. From Lemma 2 and Theorem 13 for $n = 2$, we have that

$$(2.6)\quad f(z) \leq \frac{M - z}{M - m} f(m) + \frac{z - m}{M - m} f(M) - \frac{M - z}{M - m} f(z - m) - \frac{z - m}{M - m} f(M - z)$$

and

(2.7)

$$f(M + m - z) \leq \frac{z - m}{M - m} f(m) + \frac{M - z}{M - m} f(M) - \frac{z - m}{M - m} f(M - z) - \frac{M - z}{M - m} f(z - m)$$

hold for superquadratic function f, $0 \leq m \leq z \leq M$, $m < M$.

Therefore, from (2.6) and (2.7), we have

$$(2.8)\quad f(z) + f(M + m - z) \leq f(m) + f(M) - 2 \frac{z - m}{M - m} f(M - z) - 2 \frac{M - z}{M - m} f(z - m).$$

Now for $t \in [0,1]$ and $x,\, y \in [a,b]$ we obtain from (2.8) the following inequalities:

By setting $z = t\frac{x+a}{2} + (1-t)\,\frac{a+b}{2}$, $M = t\frac{x+a}{2} + (1-t)\,\frac{a+2b-y}{2}$, $m = t\frac{x+a}{2} + (1-t)\,\frac{y+a}{2}$ in (2.8), we obtain that

$$\frac{1}{2}f\left(t\frac{x+a}{2} + (1-t)\,\frac{a+b}{2}\right) \leq \frac{1}{4}\left[f\left(t\frac{x+a}{2} + (1-t)\,\frac{y+a}{2}\right) + f\left(t\frac{x+a}{2} + (1-t)\,\frac{a+2b-y}{2}\right)\right] - \frac{1}{2}f\left((1-t)\,\frac{b-y}{2}\right) \tag{2.9}$$

holds.

Also, by replacing $z = t\frac{x+b}{2} + (1-t)\,\frac{a+b}{2}$, $M = t\frac{x+b}{2} + (1-t)\,\frac{a+2b-y}{2}$, $m = t\frac{x+b}{2} + (1-t)\,\frac{y+a}{2}$ in (2.8), we observe that

$$\frac{1}{2}f\left(t\frac{x+b}{2} + (1-t)\,\frac{a+b}{2}\right) \leq \frac{1}{4}\left[f\left(t\frac{x+b}{2} + (1-t)\,\frac{y+a}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\,\frac{a+2b-y}{2}\right)\right] - \frac{1}{2}f\left((1-t)\,\frac{b-y}{2}\right) \tag{2.10}$$

holds.

Before we proceed to our first result we quote a very useful result from [14].

Theorem 2.4. [14, Theorem 10, p.5] *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I be defined as above, then for $0 \leq s \leq t \leq 1$, $t > 0$, we have the following inequality:*

$$I(s) \leq I(t) - \int_a^b \frac{t+s}{2t} f\left(\frac{t-s}{2}\,(b-x)\right) p\,(x)\, dx - \int_a^b \frac{t-s}{2t} f\left(\frac{t+s}{2}\,(b-x)\right) p\,(x)\, dx. \tag{2.11}$$

Theorem 2.5. *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I, K be defined as above, then we have the following inequalities:*

$$I(t) \int_a^b p\,(x)\, dx \leq K(t) - \int_a^b f\left((1-t)\,\frac{b-x}{2}\right) p\,(x)\, dx \int_a^b p\,(x)\, dx \tag{2.12}$$

for all $t \in [0,1]$ *and*

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2 \leq K\left(\frac{1}{2}\right) - 2\int_a^b f\left(\frac{1}{4}(b-x)\right)p(x)\,dx\int_a^b p(x)\,dx. \tag{2.13}$$

Proof. Using the substitution rules for integration and by the assumptions on p, we have that the following identity:

$$\begin{aligned} K(t) = \frac{1}{4}\int_a^b\int_a^b \Bigg[& f\left(t\frac{x+a}{2} + (1-t)\frac{y+a}{2}\right) + f\left(t\frac{x+a}{2} + (1-t)\frac{a+2b-y}{2}\right) \\ & + f\left(t\frac{x+b}{2} + (1-t)\frac{y+a}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\frac{a+2b-y}{2}\right)\Bigg] p(x)p(y)dxdy, \end{aligned} \tag{2.14}$$

holds for all $t \in [0,1]$.

By adding (2.9) and (2.10) and then integrating the obtained result over x on $[a,b]$ and over y on $[a,b]$, we derive the inequality (2.12).

Now from (2.11), for $s = 0$, we have that

$$I(0) = f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leq I(t) - \int_a^b f\left(\frac{t}{2}(b-x)\right)p(x)\,dx,$$

from which we get by setting $t = \frac{1}{2}$ that

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2 \leq \left[I\left(\frac{1}{2}\right) - \int_a^b f\left(\frac{1}{4}(b-x)\right)p(x)\,dx\right]\int_a^b p(x)\,dx. \tag{2.15}$$

For $t = \frac{1}{2}$, (2.12) gives us the following inequality:

$$I\left(\frac{1}{2}\right)\int_a^b p(x)\,dx \leq K\left(\frac{1}{2}\right) - \int_a^b f\left(\frac{1}{4}(b-x)\right)p(x)\,dx\int_a^b p(x)\,dx. \tag{2.16}$$

From (2.15) and (2.16), we derive (2.13).

This completes the proof. □

Remark 1. When superquadratic function f is non-negative and hence convex, then the inequalities (2.12) and (2.13) represent refinements of the inequalities (1.5) and (1.6) in Theorem 1 respectively.

Remark 2. (a) From (2.11), we get that

$$I(0)\int_a^b p(x)\,dx+\int_a^b f\left(\frac{t}{2}(b-x)\right)p(x)\,dx\int_a^b p(x)\,dx\le I(t)\int_a^b p(x)\,dx$$
$$-\int_a^b\left[\frac{1-t}{2}f\left(\frac{1+t}{2}(b-x)\right)-\frac{1+t}{2}f\left(\frac{1-t}{2}(b-x)\right)\right]p(x)\,dx\int_a^b p(x)\,dx. \tag{2.17}$$

Therefore from (2.12) and (2.17), we have

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2+\int_a^b f\left(\frac{t}{2}(b-x)\right)p(x)\,dx\int_a^b p(x)\,dx$$
$$\le I(t)\int_a^b p(x)\,dx\le K(t)-\int_a^b f\left((1-t)\frac{b-x}{2}\right)p(x)\,dx\int_a^b p(x)\,dx$$
$$\le\int_a^b\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]p(x)dx\int_a^b p(x)dx$$
$$-\int_a^b\left[\frac{1-t}{2}f\left(\frac{1+t}{2}(b-x)\right)-\frac{1+t}{2}f\left(\frac{1-t}{2}(b-x)\right)\right]p(x)\,dx\int_a^b p(x)\,dx. \tag{2.18}$$

The inequalities in (2.18) represent a refinement of the inequalities given in [20, Remark 5, p.6], when superquadratic function f is non-negative and hence convex.

(b) Let $p(x)=\frac{1}{b-a}$, $x\in[a,b]$ in Theorem 15, then $I(t)=H(t)$ and $K(t)=F(t)$, $t\in[0,1]$, therefore the inequalities (2.12) and (2.13) take the form

$$H(t)\le F(t)-\frac{1}{b-a}\int_a^b f\left((1-t)\frac{b-x}{2}\right)dx, \tag{2.19}$$

for all $t\in[0,1]$ and

$$f\left(\frac{a+b}{2}\right)\le F\left(\frac{1}{2}\right)-\frac{2}{b-a}\int_a^b f\left(\frac{1}{4}(b-x)\right)dx. \tag{2.20}$$

The inequalities (2.19) and (2.20) represent refinements of the inequalities (1.5) in [20, Theorem D, p.2-3] when superquadratic function f is non-negative and hence convex.

Now we again go through some calculations like those given before Theorem 15 to give our next result.

The following inequalities hold for all $t \in [0,1]$, $x \in [a,b]$ and $y \in \left[a, \frac{3a+b}{4}\right]$ for superquadratic function f:

By setting $z = t\frac{x+a}{2} + (1-t)\,y$, $M = t\frac{x+a}{2} + (1-t)\,\frac{a+b}{2}$, $m = t\frac{x+a}{2} + (1-t)\,a$ in (2.8), we obtain that

$$\begin{aligned}(2.21)\quad & f\left(t\frac{x+a}{2} + (1-t)\,y\right) + f\left(t\frac{x+a}{2} + (1-t)\left(\frac{3a+b}{2} - y\right)\right) \\ & \le f\left(t\frac{x+a}{2} + (1-t)\,a\right) + f\left(t\frac{x+a}{2} + (1-t)\,\frac{a+b}{2}\right) \\ & - \frac{2\,(a+b-2y)}{b-a} f\left((1-t)\,(y-a)\right) - \frac{4\,(y-a)}{b-a} f\left((1-t)\left(\frac{a+b}{2} - y\right)\right)\end{aligned}$$

holds.

Also, by replacing $z = t\frac{x+a}{2} + (1-t)\left(\frac{b-a}{2} + y\right)$, $M = t\frac{x+a}{2} + (1-t)\,b$, $m = t\frac{x+a}{2} + (1-t)\,\frac{a+b}{2}$ in (2.8), we get that

$$\begin{aligned}(2.22)\quad & f\left(t\frac{x+a}{2} + (1-t)\left(\frac{b-a}{2} + y\right)\right) + f\left(t\frac{x+a}{2} + (1-t)\,(a+b-y)\right) \\ & \le f\left(t\frac{x+a}{2} + (1-t)\,\frac{a+b}{2}\right) + f\left(t\frac{x+a}{2} + (1-t)\,b\right) \\ & - \frac{2\,(a+b-2y)}{b-a} f\left((1-t)\,(y-a)\right) - \frac{4\,(y-a)}{b-a} f\left((1-t)\left(\frac{a+b}{2} - y\right)\right)\end{aligned}$$

holds.

Again, by replacing $z = t\frac{x+b}{2} + (1-t)\,y$, $M = t\frac{x+b}{2} + (1-t)\,\frac{a+b}{2}$, $m = t\frac{x+b}{2} + (1-t)\,a$ in (2.8), we observe that

$$\begin{aligned} (2.23)\quad & f\left(t\frac{x+b}{2} + (1-t)\left(\frac{b-a}{2} + y\right)\right) + f\left(t\frac{x+b}{2} + (1-t)\,(a+b-y)\right) \\ & \leq f\left(t\frac{x+b}{2} + (1-t)\,\frac{a+b}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\,b\right) \\ & - \frac{2\,(a+b-2y)}{b-a} f\left((1-t)\,(y-a)\right) - \frac{4\,(y-a)}{b-a} f\left((1-t)\left(\frac{a+b}{2} - y\right)\right) \end{aligned}$$

holds.

Finally, by replacing $z = t\frac{x+b}{2} + (1-t)\left(\frac{b-a}{2} + y\right)$, $M = t\frac{x+b}{2} + (1-t)\,b$, $m = t\frac{x+b}{2} + (1-t)\,\frac{a+b}{2}$ in (2.8), we obtain that

$$\begin{aligned} (2.24)\quad & f\left(t\frac{x+b}{2} + (1-t)\left(\frac{b-a}{2} + y\right)\right) + f\left(t\frac{x+b}{2} + (1-t)\,(a+b-y)\right) \\ & \leq f\left(t\frac{x+b}{2} + (1-t)\,\frac{a+b}{2}\right) + f\left(t\frac{x+b}{2} + (1-t)\,b\right) \\ & - \frac{2\,(a+b-2y)}{b-a} f\left((1-t)\,(y-a)\right) - \frac{4\,(y-a)}{b-a} f\left((1-t)\left(\frac{a+b}{2} - y\right)\right) \end{aligned}$$

holds.

Now we are ready to state and prove our next result as follow:

Theorem 2.6. *Let f be superquadratic integrable function on $[0, b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I, K, S_p be defined as above, then we have the following inequality:*

$$\begin{aligned} (2.25)\quad & K(t) - I(t) \int_a^b p(x)dx \leq S_p\,(1-t) \int_a^b p(x)dx - K(t) \\ & - \int_a^b \left[\frac{b-x}{b-a} f\left((1-t)\,\frac{x-a}{2}\right) + \frac{x-a}{b-a} f\left((1-t)\,\frac{b-x}{2}\right)\right] p(x)dx \int_a^b p(x)dx, \end{aligned}$$

for all $t \in [a, b]$.

Proof. Using the substitution rules for integration and by the assumptions on p, we have that the following identity:

(2.26)

$$\begin{aligned}
K(t) = \frac{1}{4}\int_a^b\int_a^b \Bigg[& f\left(t\frac{x+a}{2}+(1-t)\frac{y+a}{2}\right)+f\left(t\frac{x+a}{2}+(1-t)\frac{a+2b-y}{2}\right) \\
&+f\left(t\frac{x+b}{2}+(1-t)\frac{y+a}{2}\right)+f\left(t\frac{x+b}{2}+(1-t)\frac{a+2b-y}{2}\right)\Bigg] p(x)p(y)dxdy \\
=\frac{1}{2}\int_a^b\int_a^{\frac{a+b}{2}} \Bigg[& f\left(t\frac{x+a}{2}+(1-t)y\right)+f\left(t\frac{x+a}{2}+(1-t)(a+b-y)\right) \\
&+f\left(t\frac{x+b}{2}+(1-t)y\right)+f\left(t\frac{x+b}{2}+(1-t)(a+b-y)\right)\Bigg] p(2y-a)p(x)dydx \\
=\frac{1}{2}\int_a^b\int_a^{\frac{3a+b}{4}} \Bigg[& f\left(t\frac{x+a}{2}+(1-t)y\right)+f\left(t\frac{x+a}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \\
&+f\left(t\frac{x+a}{2}+(1-t)(a+b-y)\right)+f\left(t\frac{x+a}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right) \\
&+f\left(t\frac{x+b}{2}+(1-t)(a+b-y)\right)+f\left(t\frac{x+b}{2}+(1-t)\left(\frac{3a+b}{2}-y\right)\right) \\
&+f\left(t\frac{x+b}{2}+(1-t)y\right)+f\left(t\frac{x+b}{2}+(1-t)\left(\frac{b-a}{2}+y\right)\right)\Bigg] p(2y-a)p(x)dydx,
\end{aligned}$$

for all $t \in [a,b]$.

Multiplying (2.21)-(2.24) by $p(x)p(2y-a)$, integrating over x on $[a,b]$, over y on $\left[a, \frac{3a+b}{4}\right]$ and using the identity (2.26), we have that

$$\begin{aligned}
(2.27)\quad 2K(t) \le & \left[I(t)+S_p(1-t)\right]\int_a^b p(x)dx \\
& -4\int_a^{\frac{3a+b}{4}} \frac{a+b-2y}{b-a} f\left((1-t)(y-a)\right)p(2y-a)dx\int_a^b p(x)dx \\
& -8\int_a^{\frac{3a+b}{4}} \frac{y-a}{b-a} f\left((1-t)\left(\frac{a+b}{2}-y\right)\right)p(2y-a)dx\int_a^b p(x)dx
\end{aligned}$$

By the change of variable $y \to \frac{y+a}{2}$ and then by the change of variable $y \to a+b-y$ in the last two integrals of (2.28), under the assumptions on p, we get (2.25).

This completes the proof as well. □

Remark 3. (a) The inequalities in (2.25) represent a refinement of the inequalities in (1.7) of Theorem 2, when the superquadratic function is non-negative and hence convex.

(b) If $g(x) = \frac{1}{b-a}$, $x \in [a,b]$ in Theorem 16, then $K(t) = F(t)$, $I(t) = H(t)$, $S_p(1-t) = L(1-t)$, for all $t \in [0,1]$. Therefore Theorem 16 reduces to

$$F(t) - H(t) \leq L(1-t) - F(t) - \int_a^b \frac{b-x}{(b-a)^2} f\left((1-t)\frac{x-a}{2}\right)dx - \int_a^b \frac{x-a}{(b-a)^2} f\left((1-t)\frac{b-x}{2}\right)dx, \tag{2.28}$$

for all $t \in [0,1]$. If superquadratic function f is non-negative and therefore convex then the inequality in (2.28) represents a refinement of the inequality (1.10) from [20, Theorem I, p.4].

Before we give the last result of this section we again quote some results from [13] and [15].

Theorem 2.7. [13, Theorem 16, p.16] *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let G, L_p be defined as above, then we have the following inequality:*

$$G(t)\int_a^b p(x)\,dx \leq L_p(t) - \int_a^b f\left((1-t)\left|x - \frac{a+b}{2}\right|\right)p(x)dx, \tag{2.29}$$

for all $t \in [0,1]$.

Theorem 2.8. [15, Theorem 8, p.9] *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let I and G be defined as above, then the following inequality holds for all $t \in [0,1]$:*

$$I(t) \leq G(t)\int_a^b p(x)\,dx - \int_a^b \frac{1}{2}\left[\frac{x-a}{b-a} f\left(t\frac{2b-x-a}{2}\right) + \frac{2b-x-a}{b-a} f\left(t\frac{x-a}{2}\right)\right]p(x)dx. \tag{2.30}$$

Theorem 2.9. *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$, $0\le a<b$. Let S_p and G be defined as above, then the following inequality holds for all $t\in[0,1]$:*

$$G(t)\int_a^b p(x)dx \le S_p(t)-\int_a^b f\left((1-t)\left(\frac{b-x}{2}\right)\right)p(x)dx. \tag{2.31}$$

Now we state the last result and summarize the related results to it in a corollary followed by Theorem 20.

Theorem 2.10. *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x=\frac{a+b}{2}$, $0\le a<b$. Let I, G, S_p, L_p, K be defined as above, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2+\int_a^b f\left(\frac{t}{2}(b-x)\right)p(x)\,dx\int_a^b p(x)\,dx$$
$$\le I(t)\int_a^b p(x)\,dx \le K(t)-\int_a^b f\left((1-t)\frac{b-x}{2}\right)p(x)\,dx\int_a^b p(x)\,dx$$

$$\le \frac{1}{2}\left[I(t)+S_p(1-t)\right]\int_a^b p(x)\,dx-\frac{1}{2}\left[\int_a^b \frac{b-x}{b-a}f\left((1-t)\frac{x-a}{2}\right)p(x)dx\right.$$
$$\left.+\int_a^b \frac{x+2b-3a}{b-a}f\left((1-t)\frac{b-x}{2}\right)p(x)dx\right]\int_a^b p(x)\,dx$$

$$\le \frac{1}{2}\left[G(t)+S_p(1-t)\right]\int_a^b p(x)\,dx-\frac{1}{4}\left[\int_a^b \frac{x-a}{b-a}f\left(t\frac{2b-x-a}{2}\right)p(x)dx\right.$$
$$+\int_a^b \frac{2b-x-a}{b-a}f\left(t\frac{x-a}{2}\right)p(x)dx+2\int_a^b \frac{b-x}{b-a}f\left((1-t)\frac{x-a}{2}\right)p(x)dx$$
$$\left.+2\int_a^b \frac{x+2b-3a}{b-a}f\left((1-t)\frac{b-x}{2}\right)p(x)dx\right]\int_a^b p(x)\,dx$$

$$\le \frac{1}{2}\left[L_p(t)+S_p(1-t)\right]\int_a^b p(x)\,dx-\frac{1}{4}\left[\int_a^b \frac{x-a}{b-a}f\left(t\frac{2b-x-a}{2}\right)p(x)dx\right.$$

$$+\int_a^b \frac{2b-x-a}{b-a}f\left(t\frac{x-a}{2}\right)p(x)dx+2\int_a^b \frac{x+2b-3a}{b-a}f\left((1-t)\frac{b-x}{2}\right)p(x)dx$$

$$\left.+2\int_a^b f\left((1-t)\left|x-\frac{a+b}{2}\right|\right)p(x)dx+2\int_a^b \frac{b-x}{b-a}f\left((1-t)\frac{x-a}{2}\right)p(x)dx\right]\int_a^b p(x)\,dx$$

$$\le\left[\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]-\int_a^b \frac{1-t}{2}f\left(\frac{1+t}{2}(b-x)\right)p(x)\,dx\right.$$
$$\left.-\int_a^b \frac{1+t}{2}f\left(\frac{1-t}{2}(b-x)\right)p(x)\,dx\right]\int_a^b p(x)\,dx \tag{2.32}$$

and

$$f\left(\frac{a+b}{2}\right)\left(\int_a^b p(x)\,dx\right)^2+\int_a^b f\left(\frac{t}{2}(b-x)\right)p(x)\,dx\int_a^b p(x)\,dx$$

$$\le I(t)\int_a^b p(x)\,dx\le K(t)-\int_a^b f\left((1-t)\frac{b-x}{2}\right)p(x)\,dx\int_a^b p(x)\,dx$$

$$\le \frac{1}{2}\left[I(t)+S_p(1-t)\right]\int_a^b p(x)\,dx-\frac{1}{2}\left[\int_a^b \frac{b-x}{b-a}f\left((1-t)\frac{x-a}{2}\right)p(x)dx\right.$$

$$\left.+\int_a^b \frac{x+2b-3a}{b-a}f\left((1-t)\frac{b-x}{2}\right)p(x)dx\right]\int_a^b p(x)\,dx$$

$$\le \frac{1}{2}\left[G(t)+S_p(1-t)\right]\int_a^b p(x)\,dx-\frac{1}{4}\left[\int_a^b \frac{x-a}{b-a}f\left(t\frac{2b-x-a}{2}\right)p(x)dx\right.$$

$$+\int_a^b \frac{2b-x-a}{b-a}f\left(t\frac{x-a}{2}\right)p(x)dx+2\int_a^b \frac{b-x}{b-a}f\left((1-t)\frac{x-a}{2}\right)p(x)dx$$

$$\left.+2\int_a^b \frac{x+2b-3a}{b-a}f\left((1-t)\frac{b-x}{2}\right)p(x)dx\right]\int_a^b p(x)\,dx$$

$$\leq \frac{1}{2}\left[S_p(t)+S_p(1-t)\right]\int_a^b p(x)\,dx - \frac{1}{4}\left[\int_a^b \frac{x-a}{b-a} f\left(t\frac{2b-x-a}{2}\right)p(x)dx\right.$$

$$+\int_a^b \frac{2b-x-a}{b-a} f\left(t\frac{x-a}{2}\right)p(x)dx + 2\int_a^b \frac{b-x}{b-a} f\left((1-t)\frac{x-a}{2}\right)p(x)dx$$

$$\left.+2\int_a^b \frac{x+3b-4a}{b-a} f\left((1-t)\frac{b-x}{2}\right)p(x)dx\right]\int_a^b p(x)\,dx$$

$$(2.33)\quad \leq \left[\frac{1}{2}\left\{f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right\} - \int_a^b \frac{1-t}{2} f\left(\frac{1+t}{2}(b-x)\right)p(x)\,dx\right.$$

$$\left.-\int_a^b \frac{1+t}{2} f\left(\frac{1-t}{2}(b-x)\right)p(x)\,dx\right]\int_a^b p(x)\,dx,$$

for all $t \in [0,1]$.

Proof. This a direct consequence of Theorem 14, Theorem 15, Theorem 16, Theorem 17, Theorem 18 and Theorem 20. □

Remark 4. The inequalities (2.32) and (2.33) in Theorem 20 represent a refinement of the inequalities (1.13) and (1.14) in Theorem 10 when the superquadratic function f is non-negative and hence convex.

If we take $p(x) = \frac{1}{b-a}$, $x \in [a,b]$, then we have the following corollary which is a natural consequence of Theorem 20:

Corollary 2.1. *Let f be superquadratic integrable function on $[0,b]$ and $p(x)$ be non-negative integrable and symmetric about $x = \frac{a+b}{2}$, $0 \leq a < b$. Let G, H, F, L be defined as above and let $p(x) = \frac{1}{b-a}$, $x \in [a,b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b f\left(\frac{t}{2}(b-x)\right)dx$$

$$\leq H(t) \leq F(t) - \frac{1}{b-a}\int_a^b f\left((1-t)\frac{b-x}{2}\right)dx$$

$$\le \frac{1}{2}\left[H(t)+L(1-t)\right]-\frac{1}{2}\int_a^b \frac{b-x}{(b-a)^2} f\left((1-t)\frac{x-a}{2}\right)dx$$
$$-\frac{1}{2}\int_a^b \frac{x+2b-3a}{(b-a)^2} f\left((1-t)\frac{b-x}{2}\right)dx$$

$$\le \frac{1}{2}\left[G(t)+L(1-t)\right]-\frac{1}{4}\int_a^b \frac{x-a}{(b-a)^2} f\left(t\frac{2b-x-a}{2}\right)dx$$
$$-\frac{1}{4}\int_a^b \frac{2b-x-a}{(b-a)^2} f\left(t\frac{x-a}{2}\right)dx-\frac{1}{2}\int_a^b \frac{b-x}{(b-a)^2} f\left((1-t)\frac{x-a}{2}\right)dx$$
$$-\frac{1}{2}\int_a^b \frac{x+2b-3a}{(b-a)^2} f\left((1-t)\frac{b-x}{2}\right)dx$$

$$\le \frac{1}{2}\left[L(t)+L(1-t)\right]-\frac{1}{4}\int_a^b \frac{x-a}{(b-a)^2} f\left(t\frac{2b-x-a}{2}\right)dx$$
$$-\frac{1}{4}\int_a^b \frac{2b-x-a}{(b-a)^2} f\left(t\frac{x-a}{2}\right)dx-\frac{1}{2}\int_a^b \frac{x+3b-4a}{(b-a)^2} f\left((1-t)\frac{b-x}{2}\right)dx$$
$$-\frac{1}{2}\int_a^b \frac{b-x}{(b-a)^2} f\left((1-t)\frac{x-a}{2}\right)dx$$

$$\le \frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+a}{2}\right)\right]-\frac{1}{b-a}\int_a^b \frac{1-t}{2} f\left(\frac{1+t}{2}(b-x)\right)dx$$
$$-\frac{1}{b-a}\int_a^b \frac{1+t}{2} f\left(\frac{1-t}{2}(b-x)\right)dx, \tag{2.34}$$

for all $t\in[0,1]$.

Proof. Since for $p(x)=\frac{1}{b-a}$, $x\in[a,b]$, $S_p(t)=L_p(t)=L(t)$; $K(t)=F(t)$; $I(t)=H(t)$ and

$$\int_a^b f\left((1-t)\left|x-\frac{a+b}{2}\right|\right)p(x)dx=\int_a^b f\left((1-t)\frac{b-x}{2}\right)p(x)dx,$$

for all $t\in[a,b]$. Therefore proof of the corollary follows directly from Theorem 20. □

Remark 5. The inequalities in (2.34) represent a refinement of the inequalities given in [20, Corollay 9, p.10] when superquadratic function is non-negative and therefore convex.

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