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*g-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

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ABSTRACT. Characterizations and properties of $\mathcal{I}_{*g}$-closed sets and $\mathcal{I}_{*g}$-open sets are given. A characterization of normal spaces is given in terms of $\mathcal{I}_{*g}$-open sets. Also, it is established that an $\mathcal{I}_{*g}$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

1. **Introduction and preliminaries**

An ideal $\mathcal{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies (i) A$\in\mathcal{I}$ and B$\subset$A$\Rightarrow$B$\in\mathcal{I}$ and (ii) A$\in\mathcal{I}$ and B$\in\mathcal{I}\Rightarrow$A$\cup$B$\in\mathcal{I}$. Given a topological space (X,τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X)$\rightarrow\wp$(X), called a local function [8] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A$\subseteq$X, $A^*(\mathcal{I},\tau)$={x$\in$X | U$\cap$A$\notin\mathcal{I}$ for every U$\in\tau$(x)} where τ(x)={U$\in\tau$ | x$\in$U}. We will make use of the basic facts about the local functions [[7], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A)=A\cup A^*(\mathcal{I},\tau)$ [16]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$.

If $\mathcal{I}$ is an ideal on X, then (X,τ,$\mathcal{I}$) is called an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X,τ). A subset A of an ideal space (X,τ,$\mathcal{I}$) is $\star$-closed [7]

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(resp. $\star$-dense in itself [5]) if $A^*\subseteq A$ (resp. $A\subseteq A^*$). A subset A of an ideal space $(X,\tau,\mathcal{I})$ is $\mathcal{I}_g$-closed [2] if $A^*\subseteq U$ whenever $A\subseteq U$ and U is open.

By a space, we always mean a topological space (X,τ) with no separation properties assumed. If $A\subseteq X$, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X,τ) and $\text{int}^*(A)$ will denote the interior of A in (X,τ^*).

A subset A of a space (X,τ) is an α-open [14] (resp. semi-open [9], preopen [11]) set if $A\subseteq\text{int}(\text{cl}(\text{int}(A)))$ (resp. $A\subseteq\text{cl}(\text{int}(A))$, $A\subseteq\text{int}(\text{cl}(A))$).

The family of all α-open sets in (X,τ), denoted by τ^α, is a topology on X finer than τ. The closure of A in (X,τ^α) is denoted by $\text{cl}_\alpha(A)$.

Definition 1.1. A subset A of a space (X,τ) is said to be

(1) g-closed [10] if $\text{cl}(A)\subseteq U$ whenever $A\subseteq U$ and U is open,
(2) $\hat{g}$-closed [17] if $\text{cl}(A)\subseteq U$ whenever $A\subseteq U$ and U is semi-open,
(3) $\hat{g}$-open [17] if its complement is $\hat{g}$-closed,
(4) *g-closed [6] if $\text{cl}(A)\subseteq U$ whenever $A\subseteq U$ and U is $\hat{g}$-open.

The family of all $\hat{g}$-open sets in (X,τ) is a topology on X. The $\hat{g}$-closure [17] of a subset A of X, denoted by $\hat{g}\text{cl}(A)$, is defined to be the intersection of all $\hat{g}$-closed sets containing A.

Definition 1.2. An ideal $\mathcal{I}$ is said to be

(1) codense [3] or τ-boundary [13] if $\tau \cap \mathcal{I}=\{\emptyset\}$,
(2) completely codense [3] if $PO(X)\cap\mathcal{I}=\{\emptyset\}$, where PO(X) is the family of all preopen sets in (X,τ).

Lemma 1.1. *Every completely codense ideal is codense but not the converse* [3].

The following Lemmas will be useful in the sequel.

Lemma 1.2. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subseteq X$. If $A\subseteq A^*$, then $A^*=cl(A^*)=cl(A)=cl^*(A)$ [[15], Theorem 5].*

Lemma 1.3. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then $\mathcal{I}$ is codense if and only if $G\subseteq G^*$ for every semi-open set G in X [[15], Theorem 3].*

Lemma 1.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If $\mathcal{I}$ is completely codense, then $\tau^*\subseteq\tau^\alpha$ [[15], Theorem 6].*

Remark 1. If (X,τ) is a topological space, then every closed set is $\hat{g}$-closed but not conversely [[1], Theorem 2.3].

Lemma 1.5. *If $(X,\tau,\mathcal{I})$ is a T_1 ideal space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set [[12], Corollary 2.2].*

Lemma 1.6. *Every g-closed set is $\mathcal{I}_g$-closed but not conversely [[2], Theorem 2.1].*

2. $\mathcal{I}_{*g}$-closed sets

Definition 2.1. A subset A of an ideal space $(X,\tau,\mathcal{I})$ is said to be

(1) $\mathcal{I}_{*g}$-closed if $A^*\subseteq U$ whenever $A\subseteq U$ and U is $\hat{g}$-open,

(2) $\mathcal{I}_{*g}$-open if its complement is $\mathcal{I}_{*g}$-closed.

Theorem 2.1. *If $(X,\tau,\mathcal{I})$ is any ideal space, then every $\mathcal{I}_{*g}$-closed set is $\mathcal{I}_g$-closed but not conversely.*

Example 2.1. *Let $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{c\},\{a,b\}\}$ and $\mathcal{I}=\{\emptyset,\{a\}\}$. Then $\mathcal{I}_{*g}$-closed sets are $\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\}$ and $\mathcal{I}_g$-closed sets are power set of X. It is clear that $\{b\}$ is $\mathcal{I}_g$-closed but it is not $\mathcal{I}_{*g}$-closed.*

The following theorem gives characterizations of $\mathcal{I}_{*g}$-closed sets.

Theorem 2.2. *If $(X,\tau,\mathcal{I})$ is any ideal space and $A\subseteq X$, then the following are equivalent.*

(1) *A is $\mathcal{I}_{*g}$-closed,*

(2) *$cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in X,*

(3) *For all $x \in cl^*(A)$, $\hat{g}cl(\{x\}) \cap A \neq \emptyset$.*

(4) *$cl^*(A) - A$ contains no nonempty $\hat{g}$-closed set,*

(5) *$A^* - A$ contains no nonempty $\hat{g}$-closed set.*

Proof. (1)⇒(2) If A is $\mathcal{I}_{*g}$-closed, then $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in X and so $cl^*(A) = A \cup A^* \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open in X. This proves (2).

(2)⇒(3) Suppose $x \in cl^*(A)$. If $\hat{g}cl(\{x\}) \cap A = \emptyset$, then $A \subseteq X - \hat{g}cl(\{x\})$. By (2), $cl^*(A) \subseteq X - \hat{g}cl(\{x\})$, a contradiction, since $x \in cl^*(A)$.

(3)⇒(4) Suppose $F \subseteq cl^*(A) - A$, F is $\hat{g}$-closed and $x \in F$. Since $F \subseteq X - A$ and F is $\hat{g}$-closed, then $A \subseteq X - F$ and F is $\hat{g}$-closed, $\hat{g}cl(\{x\}) \cap A = \emptyset$. Since $x \in cl^*(A)$ by (3), $\hat{g}cl(\{x\}) \cap A \neq \emptyset$. Therefore $cl^*(A) - A$ contains no nonempty $\hat{g}$-closed set.

(4)⇒(5) Since $cl^*(A) - A = (A \cup A^*) - A = (A \cup A^*) \cap A^c = (A \cap A^c) \cup (A^* \cap A^c) = A^* \cap A^c = A^* - A$. Therefore $A^* - A$ contains no nonempty $\hat{g}$-closed set.

(5)⇒(1) Let $A \subseteq U$ where U is $\hat{g}$-open set. Therefore $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Therefore $A^* \cap (X - U) \subseteq A^* - A$. Since A^* is always closed set, so A^* is $\hat{g}$-closed set and so $A^* \cap (X - U)$ is a $\hat{g}$-closed set contained in $A^* - A$. Therefore $A^* \cap (X - U) = \emptyset$ and hence $A^* \subseteq U$. Therefore A is $\mathcal{I}_{*g}$-closed.

Theorem 2.3. *Every $\star$-closed set is $\mathcal{I}_{*g}$-closed but not conversely.*

Proof. Let A be a $\star$-closed, then $A^* \subseteq A$. Let $A \subset U$ where U is $\hat{g}$-open. Hence $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\hat{g}$-open. Therefore A is $\mathcal{I}_{*g}$-closed.

Example 2.2. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a,b\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\mathcal{I}_{*g}$-closed sets are $\emptyset, X, \{c\}, \{a,c\}, \{b,c\}$ and $\star$-closed sets are $\emptyset, X, \{c\}$. It is clear that $\{a,c\}$ is $\mathcal{I}_{*g}$-closed set but it is not $\star$-closed.*

Theorem 2.4. *Let $(X,\tau,\mathcal{I})$ be an ideal space. For every $A \in \mathcal{I}$, A is $\mathcal{I}_{*g}$-closed.*

Proof. Let A⊆U where U is $\hat{g}$-open set. Since A*=∅ for every A∈$\mathcal{I}$, then cl*(A)=A∪A* =A⊆U. Therefore, by Theorem 2.2, A is $\mathcal{I}_{*g}$-closed.

Theorem 2.5. *If (X,τ,$\mathcal{I}$) is an ideal space, then A^* is always $\mathcal{I}_{*g}$-closed for every subset A of X.*

Proof. Let A*⊆U where U is $\hat{g}$-open. Since $(A^*)^*$⊆A* [7], we have $(A^*)^*$ ⊆U whenever A*⊆U and U is $\hat{g}$-open. Hence A* is $\mathcal{I}_{*g}$-closed.

Theorem 2.6. *Let (X,τ,$\mathcal{I}$) be an ideal space. Then every $\mathcal{I}_{*g}$-closed, $\hat{g}$-open set is $\star$-closed set.*

Proof. Since A is $\mathcal{I}_{*g}$-closed and $\hat{g}$-open. Then A*⊆A whenever A⊆A and A is $\hat{g}$-open. Hence A is $\star$-closed.

Corollary 2.1. *If (X,τ,$\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal space and A is an $\mathcal{I}_{*g}$-closed set, then A is $\star$-closed set.*

Corollary 2.2. *Let (X,τ,$\mathcal{I}$) be an ideal space and A be an $\mathcal{I}_{*g}$-closed set. Then the following are equivalent.*

(1) *A is a $\star$-closed set,*

(2) *$cl^*(A)-A$ is a $\hat{g}$-closed set,*

(3) *A^*-A is a $\hat{g}$-closed set.*

Proof. (1)⇒(2) If A is $\star$-closed, then A*⊆A and so cl*(A)−A=(A∪A*)−A=∅. Hence cl*(A)−A is $\hat{g}$-closed set.

(2)⇒(3) Since cl*(A)−A=A*−A and so A*−A is $\hat{g}$-closed set.

(3)⇒(1) If A*−A is a $\hat{g}$-closed set, since A is $\mathcal{I}_{*g}$-closed set, by Theorem 2.2, A*−A=∅ and so A is $\star$-closed.

Theorem 2.7. *Let (X,τ,$\mathcal{I}$) be an ideal space. Then every *g-closed set is an $\mathcal{I}_{*g}$-closed set but not conversely.*

Proof. Let A be a *g-closed set. Then cl(A)⊆U whenever A⊆U and U is $\hat{g}$-open. We have cl*(A)⊆cl(A)⊆U whenever A⊆U and U is $\hat{g}$-open. Hence A is $\mathcal{I}_{*g}$-closed.

Example 2.3. *Let* $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{c\},\{a,b\}\}$ *and* $\mathcal{I}=\{\emptyset,\{a\}\}$. *Then* $\mathcal{I}_{*g}$*-closed sets are* $\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\}$ *and *g-closed sets are* $\emptyset,X,\{c\},\{a,b\}$. *It is clear that* $\{a\}$ *is* $\mathcal{I}_{*g}$*-closed set but it is not *g-closed.*

Theorem 2.8. *If* $(X,\tau,\mathcal{I})$ *is an ideal space and A is a* $\star$*-dense in itself,* $\mathcal{I}_{*g}$*-closed subset of X, then A is *g-closed.*

Proof. Suppose A is a $\star$-dense in itself, $\mathcal{I}_{*g}$-closed subset of X. Let A⊆U where U is $\hat{g}$-open. Then by Theorem 2.2 (2), cl*(A)⊆U whenever A⊆U and U is $\hat{g}$-open. Since A is $\star$-dense in itself, by Lemma 1.2, cl(A)=cl*(A). Therefore cl(A)⊆U whenever A⊆U and U is $\hat{g}$-open. Hence A is *g-closed.

Corollary 2.3. *If* $(X,\tau,\mathcal{I})$ *is any ideal space where* $\mathcal{I}=\{\emptyset\}$*, then A is* $\mathcal{I}_{*g}$*-closed if and only if A is *g-closed.*

Proof. From the fact that for $\mathcal{I}=\{\emptyset\}$, A^*=cl(A)⊇A. Therefore A is $\star$-dense in itself. Since A is $\mathcal{I}_{*g}$-closed, by Theorem 2.8, A is *g-closed.

Conversely, by Theorem 2.7, every *g-closed set is $\mathcal{I}_{*g}$-closed set.

Corollary 2.4. *If* $(X,\tau,\mathcal{I})$ *is any ideal space where* $\mathcal{I}$ *is codense and A is a semi-open,* $\mathcal{I}_{*g}$*-closed subset of X, then A is *g-closed.*

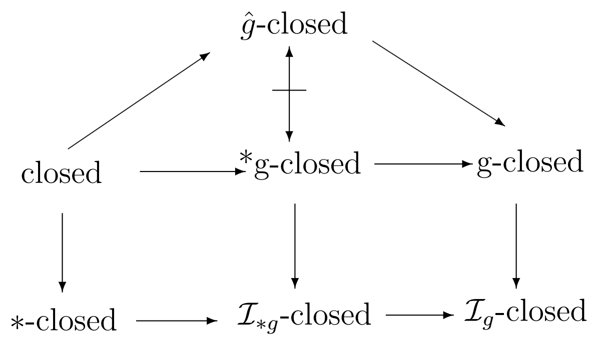
Proof. By Lemma 1.3, A is $\star$-dense in itself. By Theorem 2.8, A is *g-closed.

Example 2.4. *Let* $X=\{a,b,c\}$, $\tau=\{\emptyset,X,\{c\},\{a,b\}\}$ *and* $\mathcal{I}=\{\emptyset,\{a\}\}$. *Then g-closed sets are power set of X and* $\mathcal{I}_{*g}$*-closed sets are* $\emptyset,X,\{a\},\{c\},\{a,b\},\{a,c\}$. *It is clear that* $\{b\}$ *is g-closed set but it is not* $\mathcal{I}_{*g}$*-closed.*

Example 2.5. *Let* X=$\{a,b,c\}$, τ=$\{\emptyset,X,\{a,b\}\}$ *and* $\mathcal{I}$=$\{\emptyset,\{a\}\}$. *Then g-closed sets are* $\emptyset,X,\{c\},\{a,c\},\{b,c\}$ *and* $\mathcal{I}_{*g}$*-closed sets are* $\emptyset,X,\{a\},\{c\},\{a,c\},\{b,c\}$. *It is clear that* $\{a\}$ *is* $\mathcal{I}_{*g}$*-closed set but it is not g-closed.*

Remark 2. By Example 2.4 and Example 2.5, g-closed sets and $\mathcal{I}_{*g}$-closed sets are independent.

Remark 3. We have the following implications for the subsets stated above.



Theorem 2.9. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space and* $A \subseteq X$. *Then A is* $\mathcal{I}_{*g}$*-closed if and only if* A=$F-N$ *where F is* $\star$*-closed and N contains no nonempty* $\hat{g}$*-closed set.*

Proof. If A is $\mathcal{I}_{*g}$-closed, then by Theorem 2.2 (5), N=A^*-A contains no nonempty $\hat{g}$-closed set. If F=$cl^*(A)$, then F is $\star$-closed such that $F-N=(A \cup A^*)-(A^*-A)=(A \cup A^*) \cap (A^* \cap A^c)^c=(A \cup A^*) \cap ((A^*)^c \cup A)=(A \cup A^*) \cap (A \cup (A^*)^c)= A \cup (A^* \cap (A^*)^c)=A$.

Conversely, suppose A=F−N where F is $\star$-closed and N contains no nonempty $\hat{g}$-closed set. Let U be an $\hat{g}$-open set such that $A \subseteq U$. Then $F-N \subseteq U$ which implies that $F \cap (X-U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X-U) \subseteq F^* \cap (X-U) \subseteq F \cap (X-U) \subseteq N$. By hypothesis, since $A^* \cap (X-U)$ is $\hat{g}$-closed, $A^* \cap (X-U)=\emptyset$ and so $A^* \subseteq U$. Hence A is $\mathcal{I}_{*g}$-closed.

Theorem 2.10. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space and* $A \subseteq X$. *If* $A \subseteq B \subseteq A^*$, *then* A^*=B^* *and B is* $\star$*-dense in itself.*

Proof. Since $A\subseteq B$, then $A^*\subseteq B^*$ and since $B\subseteq A^*$, then $B^*\subseteq (A^*)^*\subseteq A^*$. Therefore $A^*=B^*$ and $B\subseteq A^*\subseteq B^*$. Hence proved.

Theorem 2.11. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A and B are subsets of X such that $A\subseteq B\subseteq cl^*(A)$ and A is $\mathcal{I}_{*g}$-closed, then B is $\mathcal{I}_{*g}$-closed.*

Proof. Since A is $\mathcal{I}_{*g}$-closed, then by Theorem 2.2 (5), $cl^*(A)-A$ contains no nonempty $\hat{g}$-closed set. Since $cl^*(B)-B\subseteq cl^*(A)-A$ and so $cl^*(B)-B$ contains no nonempty $\hat{g}$-closed set. Hence B is $\mathcal{I}_{*g}$-closed.

Corollary 2.5. *Let $(X,\tau,\mathcal{I})$ be an ideal space. If A and B are subsets of X such that $A\subseteq B\subseteq A^*$ and A is $\mathcal{I}_{*g}$-closed, then A and B are *g-closed sets.*

Proof. Let A and B be subsets of X such that $A\subseteq B\subseteq A^*$ which implies that $A\subseteq B \subseteq A^*\subseteq cl^*(A)$ and A is $\mathcal{I}_{*g}$-closed. By Theorem 2.11, B is $\mathcal{I}_{*g}$-closed. Since $A\subseteq B\subseteq A^*$, then $A^*=B^*$ and so A and B are $\star$-dense in itself. By Theorem 2.8, A and B are *g-closed.

The following theorem gives a characterization of $\mathcal{I}_{*g}$-open sets.

Theorem 2.12. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subseteq X$. Then A is $\mathcal{I}_{*g}$-open if and only if $F\subseteq int^*(A)$ whenever F is $\hat{g}$-closed and $F\subseteq A$.*

Proof. Suppose A is $\mathcal{I}_{*g}$-open. If F is $\hat{g}$-closed and $F\subseteq A$, then $X-A\subseteq X-F$ and so $cl^*(X-A)\subseteq X-F$ by Theorem 2.2 (2). Therefore $F\subseteq X-cl^*(X-A)=int^*(A)$. Hence $F\subseteq int^*(A)$.

Conversely, suppose the condition holds. Let U be a $\hat{g}$-open set such that $X-A\subseteq U$. Then $X-U\subseteq A$ and so $X-U\subseteq int^*(A)$. Therefore $cl^*(X-A)\subseteq U$. By Theorem 2.2 (2), $X-A$ is $\mathcal{I}_{*g}$-closed. Hence A is $\mathcal{I}_{*g}$-open.

Corollary 2.6. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subseteq X$. If A is $\mathcal{I}_{*g}$-open, then $F\subseteq int^*(A)$ whenever F is closed and $F\subseteq A$.*

The following theorem gives a property of $\mathcal{I}_{*g}$-closed.

Theorem 2.13. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subseteq X$. If A is $\mathcal{I}_{*g}$-open and $int^*(A)\subseteq B\subseteq A$, then B is $\mathcal{I}_{*g}$-open.*

Proof. Since A is $\mathcal{I}_{*g}$-open, then X−A is $\mathcal{I}_{*g}$-closed. By Theorem 2.2 (4), $cl^*(X-A)-(X-A)$ contains no nonempty $\hat{g}$-closed set. Since $int^*(A)\subseteq int^*(B)$ which implies that $cl^*(X-B)\subseteq cl^*(X-A)$ and so $cl^*(X-B)-(X-B)\subseteq cl^*(X-A)-(X-A)$. Hence B is $\mathcal{I}_{*g}$-open.

The following theorem gives a characterization of $\mathcal{I}_{*g}$-closed sets in terms of $\mathcal{I}_{*g}$-open sets.

Theorem 2.14. *Let $(X,\tau,\mathcal{I})$ be an ideal space and $A\subseteq X$. Then the following are equivalent.*

(1) *A is $\mathcal{I}_{*g}$-closed,*
(2) *$A\cup(X-A^*)$ is $\mathcal{I}_{*g}$-closed,*
(3) *A^*-A is $\mathcal{I}_{*g}$-open.*

Proof. (1)⇒(2) Suppose A is $\mathcal{I}_{*g}$-closed. If U is any $\hat{g}$-open set such that $A\cup(X-A^*)\subseteq U$, then $X-U\subseteq X-(A\cup(X-A^*))=X\cap(A\cup(A^*)^c)^c=A^*\cap A^c=A^*-A$. Since A is $\mathcal{I}_{*g}$-closed, by Theorem 2.2 (5), it follows that $X-U=\emptyset$ and so $X=U$. Therefore $A\cup(X-A^*)\subseteq U$ which implies that $A\cup(X-A^*)\subseteq X$ and so $(A\cup(X-A^*))^*\subseteq X^*\subseteq X=U$. Hence $A\cup(X-A^*)$ is $\mathcal{I}_{*g}$-closed.

(2)⇒(1) Suppose $A\cup(X-A^*)$ is $\mathcal{I}_{*g}$-closed. If F is any $\hat{g}$-closed set such that $F\subseteq A^*-A$, then $F\subseteq A^*$ and $F\subseteq X\setminus A$ which implies that $X-A^*\subseteq X-F$ and $A\subseteq X-F$. Therefore $A\cup(X-A^*)\subseteq A\cup(X-F)=X-F$ and $X-F$ is $\hat{g}$-open. Since $(A\cup(X-A^*))^*\subseteq X-F$ which implies that $A^*\cup(X-A^*)^*\subseteq X-F$ and so $A^*\subseteq X-F$ which implies that $F\subseteq X-A^*$. Since $F\subseteq A^*$, it follows that $F=\emptyset$. Hence A is $\mathcal{I}_{*g}$-closed.

(2)⇔(3) Since $X-(A^*-A)=X\cap(A^*\cap A^c)^c=X\cap((A^*)^c\cup A)=(X\cap (A^*)^c)\cup(X\cap A)=A\cup (X-A^*)$, the equivalence is clear.

Theorem 2.15. *Let $(X,\tau,\mathcal{I})$ be an ideal space. Then every subset of X is $\mathcal{I}_{*g}$-closed if and only if every $\hat{g}$-open set is $\star$-closed.*

Proof. Suppose every subset of X is $\mathcal{I}_{*g}$-closed. If $U\subseteq X$ is $\hat{g}$-open, then U is $\mathcal{I}_{*g}$-closed and so $U^*\subseteq U$. Hence U is $\star$-closed.

Conversely, suppose that every $\hat{g}$-open set is $\star$-closed. If U is $\hat{g}$-open set such that $A\subseteq U\subseteq X$, then $A^*\subseteq U^*\subseteq U$ and so A is $\mathcal{I}_{*g}$-closed.

The following theorem gives a characterization of normal spaces in terms of $\mathcal{I}_{*g}$-open sets.

Theorem 2.16. *Let $(X,\tau,\mathcal{I})$ be an ideal space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

1. *X is normal,*
2. *For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{*g}$-open sets U and V such that $A\subseteq U$ and $B\subseteq V$,*
3. *For any closed set A and open set V containing A, there exists an $\mathcal{I}_{*g}$-open set U such that $A\subseteq U\subseteq cl^*(U)\subseteq V$.*

Proof. (1)⇒(2) The proof follows from the fact that every open set is $\mathcal{I}_{*g}$-open.

(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and $X-V$ are disjoint closed sets, there exist disjoint $\mathcal{I}_{*g}$-open sets U and W such that $A\subseteq U$ and $X-V\subseteq W$. Since $X-V$ is $\hat{g}$-closed and W is $\mathcal{I}_{*g}$-open, $X-V\subseteq int^*(W)$ and so $X-int^*(W)\subseteq V$. Again $U\cap W=\emptyset$ which implies that $U\cap int^*(W)=\emptyset$ and so $U\subseteq X-int^*(W)$ which implies that $cl^*(U)\subseteq X-int^*(W)\subseteq V$. U is the required $\mathcal{I}_{*g}$-open sets with $A\subseteq U\subseteq cl^*(U)\subseteq V$.

(3)⇒(1) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{*g}$-open set U such that A⊆U⊆cl*(U)⊆X−B. Since U is $\mathcal{I}_{*g}$-open, A⊆int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.4, $\tau^*\subseteq\tau^\alpha$ and so int*(U) and X−cl*(U)$\in\tau^\alpha$. Hence A⊆int*(U)⊆int(cl(int(int*(U))))=G and B⊆X−cl*(U)⊆int(cl(int(X−cl*(U)))) =H. G and H are the required disjoint open sets containing A and B respectively, which proves (1).

Definition 2.2. A subset A of an ideal space (X,τ,$\mathcal{I}$) is said to be an $\alpha\hat{g}$-closed set [1] if cl_α(A)⊆U whenever A⊆U and U is $\hat{g}$-open. The complement of $\alpha\hat{g}$-closed is said to be an $\alpha\hat{g}$-open set.

If $\mathcal{I}=\mathcal{N}$, it is not difficult to see that $\mathcal{I}_{*g}$-closed sets coincide with $\alpha\hat{g}$-closed sets and so we have the following Corollary.

Corollary 2.7. *Let (X,τ,$\mathcal{I}$) be an ideal space where $\mathcal{I}=\mathcal{N}$. Then the following are equivalent.*

(1) *X is normal,*
(2) *For any disjoint closed sets A and B, there exist disjoint $\alpha\hat{g}$-open sets U and V such that A⊆U and B⊆V,*
(3) *For any closed set A and open set V containing A, there exists an $\alpha\hat{g}$-open set U such that A⊆U⊆cl_α(U)⊆V.*

Definition 2.3. A subset A of an ideal space is said to be $\mathcal{I}$-compact [4] or compact modulo $\mathcal{I}$ [13] if for every open cover $\{U_\alpha \mid \alpha\in\Delta\}$ of A, there exists a finite subset Δ_0 of Δ such that $A-\cup\{U_\alpha \mid \alpha\in\Delta_0\}\in\mathcal{I}$. The space (X,$\tau$,$\mathcal{I}$) is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.

Theorem 2.17. *Let (X,τ,$\mathcal{I}$) be an ideal space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact [[12], Theorem 2.17].*

Corollary 2.8. *Let* $(X,\tau,\mathcal{I})$ *be an ideal space. If* A *is an* $\mathcal{I}_{*g}$*-closed subset of* X*, then* A *is* $\mathcal{I}$*-compact.*

Proof. The proof follows from the fact that every $\mathcal{I}_{*g}$-closed is $\mathcal{I}_g$-closed.

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