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ON $\alpha^{\star}-$SETS AND A DECOMPOSITION THEOREM

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Abstract. We define a new family of sets in a space with a weak structure and give a decomposition of $(\omega, \omega')-$continuity, a generalization of a decomposition of continuous functions.

1. Introduction

The theory of generalized topology was studied by Császár [1] in 1997. In his various papers, properties of generalized topology, basic operators, generalized neighborhood systems, some constructions for generalized topologies, ...etc. have been introduced and studied. It is well known that generalized topology in the sense of Császár [1] is a generalization of the topology on a set. On the other hand, many important collections of sets related with topology on a set form a generalized topology. The aim of this paper is to define and study some kind subsets in generalized topological spaces and give a decomposition theorem.

A nonempty family ω of subsets of a set X is said to be a *weak structure* [5] if $\emptyset \in \omega$. By a space (X, ω), we always mean the set X with a weak structure ω. Elements of ω are called $\omega-$*open* sets and the complement of a $\omega-$open set is called

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a $\omega-$*closed* set. For $A \subset X,\ c_\omega(A)$ is the intersection of all $\omega-$closed set containing A and $i_\omega(A)$ is the union of all $\omega-$open set contained in A. A subset A of a space (X, ω) is said to be $\alpha-$open [5](resp. semiopen [5], preopen [5], $\beta-$open [5]) if $A \subset i_\omega c_\omega i_\omega(A)$ (resp. $A \subset c_\omega i_\omega(A)$, $A \subset i_\omega c_\omega(A)$, $A \subset c_\omega i_\omega c_\omega(A)$). We will denote the family of all $\alpha-$open (resp. semiopen, preopen, $\beta-$open) by α (resp. σ, π, β). A subset A is said to be $\alpha-$closed (resp. semiclosed, preclosed, $\beta-$closed) if $X - A$ is $\alpha-$open (resp. semiopen, preopen, $\beta-$open). A subset A of a space (X, ω) is said to be $\omega r-$open (resp. $\omega r-$closed, nowhere dense) if $A = i_\omega c_\omega(A)$ (resp. $A = c_\omega i_\omega(A)$, $i_\omega c_\omega(A) = \emptyset$). ω is said to be a *generalized topology* [2] if $\emptyset \in \omega$ and arbitrary union of elements of ω is again in ω. The pair (X, ω) is called a generalized space. In a generalized space (X, ω), if ω is closed under finite intersection, then (X, ω) is called a *quasi-topological space* [3]. The following lemmas will be useful in the sequel and we use Lemma 1.2 without mentioning it explicitly.

Lemma 1.1. *Let (X, ω) be a quasi-topological space. Then the following hold.*
(a) $G \cap c_\omega(A) \subset c_\omega(G \cap A)$ for every $A \subset X$ and $G \in \omega$ [4, Theorem 2.1].
(b) $i_\omega(A \cap B) = i_\omega(A) \cap i_\omega(B)$ for every subsets A and B of X [4, Theorem 2.1].

Lemma 1.2. [5] *Let (X, ω) be a space and $A,\ B \subset X$. Then the following hold.*
(a) $i_\omega(A) \subset A \subset c_\omega(A)$.
(b) $A \subset B$ implies $i_\omega(A) \subset i_\omega(B)$ and $c_\omega(A) \subset c_\omega(B)$.
(c) $i_\omega i_\omega(A) = i_\omega(A)$ and $c_\omega c_\omega(A) = c_\omega(A)$.
(d) $i_\omega(X - A) = X - c_\omega(A)$ and $c_\omega(X - A) = X - i_\omega(A)$.
(e) If $A \in \omega$, then $A = i_\omega(A)$ and if A is $\omega-$closed, then $A = c_\omega(A)$.
(f) $c_\omega i_\omega c_\omega i_\omega(A) = c_\omega i_\omega(A)$ and $i_\omega c_\omega i_\omega c_\omega(A) = i_\omega c_\omega(A)$ for every subset A of X.

2. $\omega\alpha^\star-$SETS

A subset A of a space (X, ω) is said to be a $\omega\alpha^\star-$set if $i_\omega(A) = i_\omega c_\omega i_\omega(A)$. We will denote the family of all $\omega\alpha^\star-$set in (X, ω) by $\omega\alpha^\star(X)$. If $\mathcal{M}_\omega = \cup\{A \subset$

$X \mid A \in \omega\}$, then $\mathcal{M}_\omega$ need not be in ω and for every subset A of X containing $\mathcal{M}_\omega$, $i_\omega(A) = \mathcal{M}_\omega$, $c_\omega(\mathcal{M}_\omega) = X$ and so every subset A of X containing $\mathcal{M}_\omega$ is a $\omega\alpha^{\star}-$set. In fact, X is a semiclosed set. If $X \in \omega$, then $\emptyset \in \omega\alpha^{\star}(X)$. The following Theorem 2.1 gives characterizations of $\omega\alpha^{\star}-$sets and Example 2.1 below shows that if $A \in \omega\alpha^{\star}(X)$, then $i_\omega(A)$ may be an empty set.

Theorem 2.1. *Let (X, ω) be a space and $A \subset X$. Then the following are equivalent.*
(a) $A \in \omega\alpha^{\star}(X)$.
(b) $X - A \in \beta$.
(c) $i_\omega(A)$ is $\omega r-$open.

Proof. (a)⇒(b). Suppose $A \in \omega\alpha^{\star}(X)$. Now $c_\omega i_\omega c_\omega(X - A) = X - i_\omega c_\omega i_\omega(A) = X - i_\omega(A) = c_\omega(X - A) \supset X - A$ and so $X - A \in \beta$.
(b)⇒(c). Suppose $X - A \in \beta$. Then $X - A \subset c_\omega i_\omega c_\omega(X - A)$ and so $i_\omega c_\omega i_\omega(A) \subset A$ which implies that $i_\omega c_\omega i_\omega(A) \subset i_\omega(A)$. But clearly, $i_\omega(A) \subset i_\omega c_\omega i_\omega(A)$ and so $i_\omega(A) = i_\omega c_\omega i_\omega(A)$. Hence $i_\omega(A)$ is $\omega r-$open.
That (c) implies (a) is clear. □

Example 2.1. Let $X = \mathbf{R}$, the set of all real numbers and $\omega = \{\emptyset, \{0\}, \{1\}\}$. Let $A \subset X, A \neq \emptyset$ such that $\{0, 1\} \cap A = \emptyset$. Then $i_\omega(A) = \emptyset$ and $i_\omega c_\omega i_\omega(A) = \emptyset = i_\omega(A)$. Thus $A \in \omega\alpha^{\star}(X)$ and $i_\omega(A) = \emptyset$. Also, note that if $A = \{0, 2\}$, then $i_\omega(A) = \{0\} \neq \emptyset$.

The following Theorem 2.2 below gives a property of $\omega\alpha^{\star}-$sets. Example 2.2 below shows that an $\omega\alpha^{\star}-$set need not be a $\omega r-$open set and the notions $\alpha-$open sets and $\omega\alpha^{\star}-$sets are independent.

Theorem 2.2. *Let (X, ω) be a space and $A \subset X$. Then $A \in \omega\alpha^{\star}(X)$ and A is $\alpha-$open if and only if A is $\omega r-$open.*

Proof. Suppose $A \in \omega\alpha^\star(X)$ and A is an $\alpha-$open set. Then $i_\omega(A) = i_\omega c_\omega i_\omega(A)$ and $A \subset i_\omega c_\omega i_\omega(A)$ and so $A = i_\omega(A)$. By Theorem 2.1, $A \in \omega\alpha^\star(X)$ implies that $i_\omega(A)$ is $\omega r-$open and so A is $\omega r-$open. The converse follows from Lemma 1.2(c). □

Example 2.2. (a) Consider the space of Example 2.1. If $A \subset X,\ A \neq \emptyset$ such that $0, 1 \notin A$, then $A \in \omega\alpha^\star(X)$ but A is not an $\alpha-$open set. In particular, $\mathbf{R} - \{0, 1\} \in \omega\alpha^\star(X)$ but $\mathbf{R} - \{0, 1\}$ is not $\omega r-$open.

(b) Let $X = \{a, b, c\}$ and $\omega = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}\ \}$. If $A = \{a, b\}$, then $i_\omega c_\omega i_\omega(A) = X \supset A$ and so A is an $\alpha-$open set which is not a $\omega\alpha^\star-$set. This also shows that a $\omega-$open set need not be a $\omega\alpha^\star-$set and $\emptyset \in \omega\alpha^\star(X)$ does not imply that $X \in \omega$.

The following Theorem 2.3 shows that every semiclosed set is a $\omega\alpha^\star-$set, the easy proof of which is omitted and the Example 2.3 below shows that the converse is not true. Theorem 2.4 gives characterizations of $\omega r-$open sets.

Theorem 2.3. *Let (X, ω) be a space and $A \subset X$. If A is a semiclosed set, then $A \in \omega\alpha^\star(X)$.*

Example 2.3. Consider the space in Example 2.2(b). If $A = \{a, c\}$, then $i_\omega c_\omega(A) = X$ and so A is not semiclosed. Moreover, $i_\omega c_\omega i_\omega(A) = \{a\} = i_\omega(A)$ and so A is a $\omega\alpha^\star-$set. Note that, A is preopen and not $\omega r-$open and so a preopen, $\omega\alpha^\star-$set need not be a $\omega r-$open set.

Theorem 2.4. *Let (X, ω) be a space and $A \subset X$. Then the following are equivalent.*

(a) A is a semiclosed, $\alpha-$open set.

(b) $A \in \omega\alpha^\star(X)$ and A is $\alpha-$open.

(c) $X - A \in \beta$ and A is $\alpha-$open.

(d) A is $\omega r-$open.

(e) A is semiclosed and preopen.

Proof. (a)$\Rightarrow$(b) follows from Theorem 2.3.

(b)$\Rightarrow$(c) and (c)$\Rightarrow$(d) follow from Theorem 2.1.

(d)$\Rightarrow$(a) and (d)$\Leftrightarrow$(e) are clear. □

Theorem 2.5. *Let (X,ω) be a space and $A \subset X$. Then a semiopen set $A \in \omega\alpha^\star(X)$ if and only if A is semiclosed.*

Proof. If A is semiclosed, by Theorem 2.3, $A \in \omega\alpha^\star(X)$. Conversely, suppose $A \in \omega\alpha^\star(X)$ and A is semiopen. Then $i_\omega(A) = i_\omega c_\omega i_\omega(A)$ and $c_\omega(A) = c_\omega i_\omega(A)$, since A is semiopen if and only if $c_\omega(A) = c_\omega i_\omega(A)$. Hence $i_\omega c_\omega(A) = i_\omega c_\omega i_\omega(A) = i_\omega(A) \subset A$. Therefore, A is semiclosed. □

One can easily prove that the *arbitrary union of $\beta-$open sets is a $\beta-$open set* and so by Theorem 2.1, the arbitrary intersection of $\omega\alpha^\star-$sets is a $\omega\alpha^\star-$set. The following Example 2.4 shows that the union of two $\omega\alpha^\star-$sets is not a $\omega\alpha^\star-$set.

Example 2.4. Consider the space in Example 2.2(b). If $A = \{a\}$ and $B = \{b\}$, then A and B are $\omega\alpha^\star-$sets but $A \cup B = \{a, b\}$ is not a $\omega\alpha^\star-$set.

A subset A of a space (X,ω) is said to be a $C_\omega-$*set*(resp., $B_\omega-$*set*) if there exist $U \in \omega$ and $B \in \omega\alpha^\star(X)$ (resp., B is semiclosed) such that $A = U \cap B$. The family of $C_\omega-$sets (resp., $B_\omega-$sets) in (X,ω) is denoted by $C_\omega(X)$ (resp., $B_\omega(X)$). The following Theorem 2.6 gives some properties of $C_\omega-$sets and $B_\omega-$sets. The following Example 2.5 shows that the inclusions in Theorem 2.6 (a), (b) are proper, it also shows that the condition $X \in \omega$ is essential in Theorem 2.6 (c), (d).

Theorem 2.6. *Let (X,ω) be a space. Then the following hold.*

(a) $B_\omega(X) \subset C_\omega(X)$.

(b) $\omega \subset B_\omega(X)$.

(c) If $X \in \omega$, then $\omega\alpha^\star(X) \subset C_\omega(X)$.

(d) If $X \in \omega$ and $A \subset X$ is semiclosed, then $A \in B_\omega(X)$.

Proof. (a) The proof follows from Theorem 2.3.
(b) The proof follows from the fact that X is semiclosed.
The proof of (c) and (d) are clear. □

Example 2.5. (a) Consider the space in Example 2.2(b). In Example 2.4, it is established that $\{b\}$ is a $\omega\alpha^\star-$set but it is not a semiclosed set. Therefore, $\{b\} = \{b, c\} \cap \{b\}$ is a $C_\omega-$set but not a $B_\omega-$set.
(b) Consider the space in Example 2.2(b). The set $A = \{c\} = \{c\} \cap \{c, b\}$ and A is semiclosed which implies that $A \in B_\omega(X)$ but $A \notin \omega$.
(c) The set $A = \{a, c\}$ of Example 2.3 is a $\omega\alpha^\star-$set but not a $C_\omega-$set and so the condition $X \in \omega$ cannot be dropped in Theorem 2.6(c).
(d) Consider the space in Example 2.2(b). X is semiclosed but $X \notin B_\omega(X)$. Hence the condition $X \in \omega$ cannot be dropped in Theorem 2.6(d).

The following Theorem 2.7 gives a decomposition of $\omega-$open sets in a quasi-topological space. Example 2.6 below shows that the notions $\alpha-$open set and $C_\omega-$set are independent.

Theorem 2.7. *Let (X, ω) be a space and $A \subset X$. Then the following hold.*
(a) If $A \in \omega$, then A is an $\alpha-$open set and a $C_\omega-$set.
(b) If (X, ω) is a quasi-topological space, then the converse of (a) holds.

Proof. (a) If $A \in \omega$, then clearly A is $\alpha-$open. By Theorem 2.6, $A \in C_\omega(X)$.
(b) Suppose A is an $\alpha-$open set and a $C_\omega-$set. Then $A \subset i_\omega c_\omega i_\omega(A)$ and there exists $U \in \omega$ and $B \in \omega\alpha^\star(X)$ such that $A = U \cap B$. Then $A \subset i_\omega c_\omega i_\omega(A) = i_\omega c_\omega i_\omega(U \cap B) \subset i_\omega c_\omega i_\omega(B) = i_\omega(B)$. Therefore, $A = U \cap A \subset U \cap i_\omega(B) \subset A$ which implies that $A = U \cap i_\omega(B)$. Since ω is a generalized topology, $i_\omega(B) \in \omega$ and since ω is a quasi-topology, $U \cap i_\omega(B) = A \in \omega$. □

Example 2.6. Consider the space of Example 2.1. If $A = \{0, 1\}$, then A is $\omega r-$open and hence an $\alpha-$open set but A is not a $C_\omega-$set. In Example 2.2(b), $\{a, c\}$ is a

$\omega\alpha^\star-$set by Example 2.3 and so $\{c\} = \{a, c\} \cap \{b, c\}$ is a $C_\omega-$set but $\{c\}$ is not an $\alpha-$open set.

Theorem 2.8. *Let (X, ω) be a quasi-topological space and $A \subset X$. Then $A \in \alpha$ if and only if $A = U - B$ where $U \in \omega$ and B is nowhere dense.*

Proof. Suppose $A \in \alpha$. Then $A \subset i_\omega c_\omega i_\omega(A) = B$, say. Now $i_\omega c_\omega(B - A) = i_\omega c_\omega(B \cap (X - A)) \subset i_\omega c_\omega(B) \cap i_\omega c_\omega(X - A) = i_\omega c_\omega i_\omega(A) \cap (X - c_\omega i_\omega(A)) \subset i_\omega c_\omega i_\omega(A) \cap (X - i_\omega c_\omega i_\omega(A)) = \emptyset$ and so $B - A$ is nowhere dense. Since ω is a generalized topology, $B \in \omega$ such that $A = B - (B - A)$ where $B - A$ is nowhere dense. Conversely, suppose $A = U - B$ where $U \in \omega$ and B is nowhere dense. Then $i_\omega c_\omega i_\omega(A) = i_\omega c_\omega i_\omega(U - B) = i_\omega c_\omega i_\omega(U \cap (X - B)) = i_\omega c_\omega(U \cap i_\omega(X - B)) \supset i_\omega(U \cap c_\omega i_\omega(X - B))$, by Lemma 1.1 and so $i_\omega c_\omega i_\omega(A) \supset i_\omega(U \cap (X - i_\omega c_\omega(B))) = i_\omega(U \cap X) = U \supset U - B = A$. Therefore, $A \in \alpha$. $\square$

Let (X, μ) and (Y, μ') be generalized topological spaces. A function $f : X \to Y$ is said to be $(\mu, \mu')-$continuous [2] if the inverse image of every $\mu'-$open set is a $\mu-$open set. In the same steps, we define $(\omega, \omega')-$continuity as follows. Let (X, ω) and (Y, ω') be spaces. A function $f : X \to Y$ is said to be $(\omega, \omega')-$continuous (resp., $(\alpha, \omega')-$continuous, $(C_\omega, \omega')-$continuous) if the inverse image of every $\omega'-$open set is a $\omega-$open (resp., $\alpha-$open, $C_\omega-$) set. The following Theorem 2.9 gives a decomposition of $(\omega, \omega')-$continuous functions, the proof of which follows from Theorem 2.8.

Theorem 2.9. *Let (X, ω) and (Y, ω') be spaces where (X, ω) is a quasi-topological space and $f : X \to Y$ be a function. Then the following are equivalent.*
(a) f is $(\omega, \omega')-$continuous.
(b) f is $(\alpha, \omega')-$continuous and f is $(C_\omega, \omega')-$continuous.

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