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ON ALMOST WN-INJECTIVE RINGS

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ABSTRACT: Let R be a ring. Let M_R be a module with $S = End(M_R)$ .The module M is called almost Wnil-injective (briefly right AWN-injective) if , for any $0 \neq a \in N(R)$,there exists $n \geq 1$ and an S-submodule X_a of M such that $a^n \neq 0$ and $l_M(r_R(a^n)) = Ma^n \oplus X_{a^n}$ as left S-modules .If R_R is almost Wnil-injective , then we call R is right almost Wnil-injective ring . In this paper ,we give some characterization and properties of almost Wnil-injective rings .In particular , Conditions under which right almost Wnil-injective rings are n-regular rings and n-weakly regular rings are given .Also we study rings whose simple singular right R-module are almost Wnil-injective , It is proved that if R is a NCI ring ,MC2 , whose every simple singular R-module is almost Wnil-injective , Then R is reduced .

1. INTRODUCTION

Throughout the paper R is an associative ring with identity, and is a right R-module with $S = End(M_R)$ .For $a \in R$, $r(a), l(a)$ denote the right annihilator and left annihilator of a, respectively. We write $J(R), Z(R)(Y(R))$, for the Jacobson radical and the left (right) singular ideal of R, respectively . $X \leq M$ denote that X is a submodule of M .

Following [9] a ring R is called a right (left) NPP if for aR is projective for all $a \in N(R)$ (the set of nilpotent elements) .Clearly, right (left) PP ring (that is if every principal right ideal of R is projective as right R-module) is right (left)NPP, but the converse is not true by [9]. The ring R is said to be reduced if R has no non zero nilpotent

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element. The ring R is called right (left) SXM [10], if for each $0 \neq a \in R$, $r(a) = r(a^n)$ [$l(a) = l(a^n)$] for all positive integer n satisfying $a^n \neq 0$. For example ,reduced rings are right (left)SXM ring . R is said to be Von Neumann regular (or just regular), $a \in aRa$ for every $a \in R$ [15], a ring R is called n-regular [9] if $a \in aRa$ for all $a \in N(R)$. Clearly, Von Neumann regular ring are n-regular, but the converse is not true. A ring R is called right (left) n-weakly regular if $a \in aRaR$ ($a \in RaRa$), for all $a \in N(R)$ [4].Call a ring R right MC2 if for right minimal element $k \in R$, kR is a summand in R_R, whenever kR is projective as right R-module[8]. A ring R is called weakly reversible if $ab = 0$ implies that $Rbra$ is a nil left ideal of R for all $a, b, r \in R$ [14]. Generalizations of injectivity have been discussed in many papers see [5] ,[6] . A right R-module M is called principal injective (or P-injective) , if every R-homomorphism from a principal right ideal of R to M can be extended to an R-homomorphism from R to M .

Equivalently, $l_M r_R(a) = Ma$ for all $a \in R$ [2] .In [5] ,Nicholson and Yousif studied the structure of principally injective rings and give some applications . They also continued to study rings with some other kind of injectivity , namely , GP-injective rings [6] and [10] . A ring R is called GP-injective if for any $a \in R$ there exists a positive integer n with $a^n \neq 0$ and $lr(a^n) = Ra^n$, Right GP-injective rings are called right YJ-injective rings by several authors. In [18] , Zhao introduced an almost P-injective module . Let M_R be a right R-module with $S = End(M_R)$. The module M is called AP-injective, if for any $a \in R$, there exists a left S-submodule X_a of M_R such that $l_M r_R(a) = Ma \oplus X_a$.

AP-injectivity has been generally studied (see [6]). In [9], Wei and Jianhua first introduced and characterized a right nil-injective ring, and give many properties .A ring R is said to be reversible if $ab = 0$ implies that $ba = 0$ for all $a, b \in R$. A ring R is called right nil-injective , if $a \in N(R)$, $lr(a) = Ra$. In [19], Zhao and Du introduced an almost nil-injective module. Let M_R be a module with $S = End(M_R)$. The module M is called right almost nil-injective if for any $k \in N(R)$,there exists an S-submodule X_k of M such that $l_M r_R(k) = Mk \oplus X_k$ as left S-module. If R_R is almost nil-injective then we call R a right almost nil-injective ring.

2. Characterizations of Almost Wn-Injective

In this section we introduced the notion of a right GNNP and almost WN-injective with some of their basic properties; we also give necessary and sufficient conditions for almost WN-injective to be n-regular.

Following [9] a right R-module M is called Wnil-injective , if for any $0 \neq a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism $f : a^n R \to M$ can be extends to $R \to M$ .Equivalently , if for any $0 \neq a \in N(R)$ there exists a positive integer n such that $a^n \neq 0$ and $Ra^n = lr(a^n)$.

Clearly right nil-injective module are all Wnil-injective module .

Remark [6]:

We fix the following notation .If N is a submodule of M, we write N/M to indicate that N is a direct summand of M .For an (R,R)-bimodule M, we let $R\alpha M$ be the trivial extension of R and M ,i.e , $R\alpha M = R \oplus M$ as an abelian group , with the following multiplication : (r,x) (s,y) = (rs,ry+xs)

Example 6:

A non commutative right almost nil-injective ring which is not a right Wnil-injective.

Let C be a noncommutative division subring of a division ring D such that the C-vector space ${}_C D$ has dimension >1 .Let $R = C\alpha D$ be the trivial extension of C and the C-module D. Then R is not commutative. Let $0 \neq a = (c,d) \in N(R)$ .If $c \neq 0$ .then a is invertible in R and so we can let $X_a = (0)$, If $c = 0$,then $lr(a) = (0)\alpha D$ and $Ra = (0)\alpha Cd$ .Write $D = Cd \oplus D_1$ as a left C-vector space and let $X_a = (0)\alpha D_1$ .Then

$lr(a) = Ra \oplus X_a$ .Therefore, R is right almost nil-injective. Note that $a^2 = 0$ and $lr(a) \neq Ra$. Thus R is not right W nil-injective.

Lemma 2.1 [11]:

The following conditions are equivalent for a ring R :

1- R is n-regular .

2- Every right R-module is Wnil-injective .

3- Every cyclic right R-module is Wnil-injective .

4- R is right Wnil-injective and NPP ring .

Lemma 2.2 [18]:

Suppose M is a right R-module with $S = End(M_R)$ .If $l_M r_R(a) = Ma \oplus X_a$, where X_a is a left S-submodule of M_R . Set $f : aR \to M$ is a right R-homomorphism , then $f(a) = ma + x$ with $m \in M$, $x \in X_a$.

Now we give the following definition.

Definition 2.3:

A ring R is said to be right (left) GNPP if $a^n R, (Ra^n)$ is projective for all $a \in N(R)$ and for some positive integer n , $a^n \neq 0$.

Clearly every n-regular rings , reduced rings and NPP are right GNPP rings .

Lemma 2.4 [9]:

If R is a right NPP ring ,then $Y(R) = 0$.

As a parallel result to Lemma (2.4) ,the following result was obtained :

Proposition 2.5:

Let R be a right GNPP ring .Then $Y(R) = 0$.

Proof :

Let $0 \neq a \in Y(R)$, with $a^2 = 0$ .Then $a \in N(R)$ .Since R is a right GNPP ring , then there exists a positive integer n such that $a^n \neq 0$, $a^n R$ is projective . But $a^2 = 0$,so n=1 and aR is projective .Thus $r(a)$ is a direct summand of R as a right R-module .But $a \in Y(R)$, $r(a)$ must be essential in R, which is a contradiction . Hence $Y(R) = 0$.

According to [16] ,a ring R is right GQ-injective if for any right ideal I isomorphic to a complement right ideal of R ,every right R-homomorphism of I into R extends to an endomorphism of ${}_R R$.

In [16], shows that if R is right (left) GQ-injective, then $J(R) = Y(R)$ ($J = Z$), R/J is regular.

Every regular ring is right (left) GQ-injective . Clearly , R is regular if and only if R is right (left) GQ-injective right non singular [12] .

Corollary 2.6:

If R is a right GNPP-ring ,then R is regular if and only if R is right GQ-injective .

Proof :

Since R is right GQ-injective then $Y(R) = J(R)$ and R/J is regular ring .By Proposition (2.5) $0 = Y(R) = J(R)$ .So R is regular ring .

Conversely: It is clear.

Call a ring is right NC2 if aR projective implies $aR = eR$, $e = e^2 \in R$ for all $a \in N(R)$ [11]. Every n-regular rings is NPP and NC2 rings [10].

Proposition 2.7:

If R is a ring with $l(a^n) \subseteq l(a)$, then R is right NC2 and GNPP if and only if R is n-regular .

Proof :

Let $a \in N(R)$. Since R is right GNPP, then $a^n R$ is projective for some positive integer n and $a^n \neq 0$ .Since R is right NC2 ring , $a^n R = eR$,

$e^2 = e \in R$ .Thus $a^n = ea^n$ implies that $a = ea$ ($l(a^n) \subseteq l(a)$) .So $e = ab$ for some $b \in R$,Hence $a = ea = aba \in aRa$ .Thus R is n-regular .

Conversely:

Let R is n-regular ring ,implies that R is NPP ring. So is GNPP and NC2 ring.

In [6], Stanley and Yiqiang introduced an almost generalized principally injective (AGP-injective) module. Let M be a right R-module with $S = End(M_R)$. The module M is called AGP-injective if , for any $0 \neq a \in R$, there exists a positive integer n and S-submodule X_a of M such that $a^n \neq 0$ and $l_M r_R(a^n) = Ma^n \oplus X_a$ as a left S-modules. Also studied right AGP-injective rings and give some characterization and properties which generalization results of [19] .

Now, we consider rings which are more general than WN-injective rings, an idea parallel to the notion of AGP-injective rings.

Definition 2.8:

Let M_R be a module with $S = End(M_R)$ .The module M is called almost Wnil-injective (briefly right AWN-injective) if , for any $0 \neq a \in N(R)$,there exists $n \geq 1$ and an S-submodule X_a of M such that $a^n \neq 0$ and $l_M(r_R(a^n)) = Ma^n \oplus X_{a^n}$ as left S-modules. If R_R is almost WN-injective, then we call R is right almost WN-injective ring.

Remark:

{right YJ-injective rings} $\subset$ {right nil-injective rings} $\subset$ {right WN-injective rings}
$\subset$ {right AWN-injective rings}

{right AN-injective rings} $\subset$ {right AWN-injective rings}

Examples [19]:

The ring Z of integers is AWN-injective which is not AGP- injective

Let Z_2 be a field, and $R=\begin{bmatrix} Z_2 & Z_2 \\ 0 & Z_2 \end{bmatrix}$, $N(R)=\begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$. Let $0 \neq u \in Z_2$,Then $lr\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & Z_2 \\ 0 & Z_2 \end{bmatrix}$ and $R\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & uZ_2 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$, Therefore $lr\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \neq R\begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix}$ and So R is not right WN-injective but R is AWN-injective $(lr(a)) = Ra \oplus X_a)$.

Let $R=\begin{bmatrix} 0 & Z_2 \\ 0 & Z_2 \end{bmatrix}$, where Z_2 is a field. Then $N(R)=\begin{bmatrix} 0 & Z_2 \\ 0 & 0 \end{bmatrix}$. Let $y=\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \in N(R)$, Then $Ry=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $lr(y)=R$. Therefore $lr(y) \neq Ry$ and so R is not WN-injective. But $lr(y)=Ry \oplus R$, So R is right AWN-injective

Lemma 2.9 [3]:

The following conditions are equivalent :

1- R is n-regular .

2- $N_1(R)=\{0 \neq x \in R : x^2 = 0\}$ is regular .

3- For any $a \in N(R)$, there exists a positive integer n such that $a^n \neq 0$ and $a^n R$ is generated by idempotent .

It is clear that any n-regular rings is AWN-injective but the converse is not true.

The following Theorem gives a partial converse.

Theorem 2.10:

Let R be a right SXM ring. Then the following conditions are equivalent :

1- R is n-regular .

2- R is a right AWN-injective right NPP-ring .

Proof :

(1) $\rightarrow$ (2) is clear by [Lemma 2.1]

$(2)\rightarrow(1)$, Let $0\neq a\in N(R)$.Since R is a right AWN-injective ,then there exists $n\geq 1$ such that $a^n\neq 0$ and $lr(a^n)=Ra^n\oplus X_a$, Since R is right NPP ring and $a^n\in N(R)$, $r(a^n)=(1-e)R$, $e^2=e\in R$. Therefore $\mathrm{Re}=lr(a^n)=Ra^n\oplus X_a$, $e=ra^n+x$,where $r\in R$, $x\in X_a$. So $a^n=a^ne=a^nra^n+a^nx$, $(1-a^nr)a^n=a^nx\in Ra^n\cap X_a=0$ and $a^n=a^nra^n$ this implies that $(1-ba^n)\in r(a^n)=r(a)$ [R is SXM] ,yielding $a=a\ ra^n$. Take $c=ra^{n-1}\in R$, hence $a=aca$ .Therefore R is n-regular .

Proposition 2.11:

Let R be a ring whose every simple right R-module is AWN-injective then :

1- $J(R)\cap Soc(R)=0$

2- $J(R)$ is a reduced ideal of R.

Proof :

If $J(R)\cap Soc(R)\neq 0$, then there exists a minimal right ideal kR of R with $kR\subseteq J(R)$. If kR is a direct summand ,then $kR=eR$ for some $0\neq e^2=e\in R$ and we get $e\in J(R)$,which is a contradiction . So that $(kR)^2=0$. Since $r(k)$ is maximal right ideal of R,then $R/r(k)$ is AWN-injective . Let $f:kR\rightarrow R/r(k)$ be defined by $f(kr)=r+r(k)$, Then f is a well defined R-homomorphism .Since $R/r(k)$ is AWN-injective $l_{R/r(k)}r(k)=r(k)k\oplus X_k$ where X_k is a left S-submodule of M. Therefore $1+r(k)=f(k)=bk+r(k)+X$ (Lemma 2.2) .Thus $1-bk+r(k)=x\in r(k)k\cap X_k=0$, $1-bk\in r(k)$.Since $k\in J(R)$,then $bk\in J(R)\subseteq r(k)$, which implies $1\in r(k)$, which is also a contradiction .Therefore $J(R)\cap Soc(R)=0$.

Let $0 \neq a \in J(R)$ such that $a^2 = 0$. Since $a \neq 0$, then there exists a maximal right ideal M of R containing $r(a)$. Thus R/M is AWN-injective, and $l_{R/M} r_R(a) = (R/M)a \oplus X_a \leq R/M$.

Let $f : aR \to R/M$ be defined by $f(ar) = r + M$. Then f is a well defined R-homomorphism. So there exists $r \in R, x \in X_a$ such that $1 + M = ra + M + x$, $1 - ra + M = x \in R/M \cap X_a = 0$. Hence $1 - ra \in M$ and so $1 \in M$, which is a contradiction .Hence $J(R)$ is reduced .

Lemma 2.12 [1]:

If $Y(R) = 0$,then SR is a maximal right quotient ring of R .Thus the maximal right quotient ring of any right nonsingular ring is regular .

Now, we have the following theorem.

Theorem 2.13:

If R is a right GNPP right AWN-injective ring, then the center of R $(C(R))$ is n-regular.

Proof :

Since $Y(R) = 0$ [Proposition 2.5] ,then there exists a right maximal quotient ring S of R such that it is regular Lemma (2.12) ,then $C(S)$ is also regular [The center of a regular ring is regular] .For any $0 \neq a \in N(C(R)) \subseteq N(C(S))$, there exists $s \in C(S)$ such that $a = asa = a^2 s = sa^2$. Thus $r(a^n) = r(a), l(a) = l(a^n)$ for any positive integer n . We Claim that a is n-regular in $N(C(R))$. Note that $a^2 \neq 0$, So there exists a positive integer m with $a^{2m} \neq 0$ such that $lr(a^{2m}) = Ra^{2m} \oplus X_{a^{2m}}$ for some left ideal $X_{a^{2m}}$ of R since R is right AWN-injective .Thus $lr(a^{2m-1}) = lr(a^{2m}) = Ra^{2m} \oplus X_{a^{2m}}$ and So $a^{2m-1} = da^{2m} + x$ for some $d \in R$ and $x \in X_{a^{2m}}$ .Then $a^{2m} = ada^{2m} + ax$ and $(1 - ad)a^{2m} = ax \in Ra^{2m} \cap X_{a^{2m}} = 0$.

Therefore $(1-ad)a^{2m}=0$ and $(1-ad)\in l(a^{2m})=l(a)$, and So $a=ada=a^2d$. Let $u=ad^2$ then $a=a^2d=a(a^2d)d=a^2ad^2=a^2u$ .Forany $x\in R$, $a^2(xu-ux)=0$ So $(xu-ux)\in r(a^2)=r(a)$, $0=a(xu-ux)=a(xad^2-ad^2x)=a^2(xd^2-d^2x)$, $(xd^2-d^2x)\in r(a^2)=r(a)$.

Thus $xu-ux=xad^2-ad^2x=a(xd^2-d^2x)=0$ .So $xu=ux$, $u\in C(R)$ and $a=aua$ Therefore $C(R)$ is n-regular .

Lemma 2.14 [7]:

If R is a semiprime ring ,then $r(a^n)=r(a)$ for any $a\in C(R)$ and a positive integer n .

Proposition 2.15:

If R is a semiprime right AWN-injective ring ,then the center $C(R)$ of R is n-regular .

Proof :

For any $0\neq a\in N(C(R))$, $Ra\cap l(a)=0$ .Since R is semiprime. Therefore, $l(a^m)=l(c)=r(c)=r(a^n)$ for any a positive integer n Lemma (2.14) .Note that $a^2=0$ because $Ra\cap l(a)=0$ .As in the proof of Theorem [2.13] , $C(R)$ is n-regular .

Proposition 2.16:

Let R be a ring ,if for any element $a\in N(R)$,there exists a positive integer n such that $r(a^n)\subseteq r(a)$ and $a^n\neq 0$ if $R/r(a^n)$ is AWN-injective , then R is n-regular ring.

Proof :

Let a be any element in $N(R)$ and let $f:a^nR\rightarrow R/r(a^n)$ be defined by $f(a^ns)=s+r(a^n)$ for all $s\in R$ and positive integer n and $a^n\neq 0$ .Then f is a well defined R-homomorphism .Since $R/r(a^n)$ is AWN-injective, $l_{R/r(a^n)}r_R(a^n)=(R/r(a^n))a^n\oplus X_{a^n}$, where X_{a^n} is a left S-submodule of $R/r(a^n)$, $(X_{a^n} \subseteq R)$, Then there exists $b \in R$ and $x \in X_{a^n}$ such that $1 + r(a^n) = f(a^n) = ba^n + r(a^n) + x$ (Lemma2.2).

Thus $1 - ba^n + r(a^n) = x \in R/r(a^n) \cap X_{a^n} = 0$, $1 - ba^n \in r(a^n) \subseteq r(a)$ implies that $a = aba^n$ .Take $c = ba^{n-1}$, Hence $a = aca$. Therefore R is n-regular ring.

Theorem 2.17:

Let R be a ring with $a^n R = aR$ for every $a \in R$ and a positive integer n, $a^n \neq 0$ .If every simple right R-module is AWN-injective, then R is right n-weakly regular ring.

Proof :

We will Show that $RaR + r(a) = R$ for any $a \in N(R)$,If $RaR + r(a) \neq R$,then there exists a maximal right ideal M of R containing $RaR + r(a)$. Then R/M is AWN-injective ,then $l_{R/M} r(a^n) = (R/M)a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$ .Let $f : a^n R \to R/M$ be defined by $f(a^n r) = r + M$ .Note that f is well defined .So $1 + M = f(a^n) = ca^n + M + x$, $c \in R$, $x \in X_{a^n}$, $1 - ca^n + M = x \in R/M \cap X = 0$.

So $1 - ca^n \in M$,Since $ca^n \in Ra^n R = RaR \subseteq M$, $1 \in M$, Which is a contradiction . Therefore $RaR + r(a) = R$ for any $a \in N(R)$, then R is a right n-weakly regular .

Following [13], a ring R is called right N duo if aR is an ideal of R for all $a \in N(R)$. Every reduced rings is N duo.

Now , we give the definition .

Definition 2.18:

An element $x \in N(R)$ is called right (left) generalized n-regular if there exists a positive integer n such that $x^n \neq 0$ and $x^n = x^n yx$ ($x^n = xyx^n$) for some $y \in R$. A ring R is called right (left) generalized n-regular if every element in $N(R)$ is right (left) generalized n-regular .

Theorem 2.19:

Let R be AWN-injective ring with $lr(a^n)=l(r(a^{n-1})$ for every $a\in N(R)$ and $a^n\neq 0$. Then R is generalized n-regular.

Proof :

Suppose that $a\in N(R)$ .Then there exists a positive integer n such that $a^n\neq 0$ and $lr(a^n)=Ra^n\oplus X$ for some $X\leq R$. Since $lr(a^n)=l(r(a^{n-1})$, then $lr(a^{n-1})=Ra^n\oplus X$ and $a^{n-1}=da^n+x$ for some $d\in R$, $x\in X$. So $a^n=ada^n+ax$, $ax=a^n-ada^n\in Ra^n\bigcap X=0$, $a^n=ada^n$.

This proves that R is generalized n-regular.

Definition 2.20:

A ring R is called right Quasi-Nduo ring if every right maximal right ideal is right Nduo.

Theorem 2.21 :

Let R be a right quasi N duo and every simple right R-module is AWN-injective Then every element of $N(R)$ is strongly Π-regular.

Proof :

For any $0\neq a\in N(R)$, we will show that there exists a positive integer n such that $a^nR+r(a^n)=R$ .Suppose not ,then there exists a maximal right ideal M of R containing $a^nR+r(a^n)$.Since R/M is AWN-injective , $l_{R/M}(r_R(a^n))=(R/M)a^n+X_{a^n}$, $X_{a^n}\leq R/M$ and $a^n\neq 0$. Let $f:a^nR\to R/M$ be defined by $f(a^nr)=r+M$ .Since $a^nR+r(a^n)\subseteq M$, f is well defined R-homomorphism .Thus there exists $c\in R$, $x\in X_{a^n}$ such that $1+M=ca^n+M+x$, by Lemma (2.2) ,then $1-ca^n+M=x\in(R/M)a^n\cap X_{a^n}=0$,

$1-ca^n \in M$ and $ca^n \in M$ (R is right N duo) and So $1 \in M$,which is a contradiction. Therefore $a^n R + r(a^n) = R$ .In particular $a^n x + y = 1$, $x \in R$, $y \in r(a^n)$,So $a^n = a^{2n} + x$. Thus a is strongly Π-regular.

3- On Simple Singular AWN-injective Modules

In this section ,we study of rings whose Simple singular right R-module are AWN-injective .Also we give the relation between this rings and reduced rings .

A right MC2 ring R is called strongly right MC2 if R is also weakly reversible ring [12].

Now , the following result is given :

Proposition 3.1:

Let R be a ring whose every simple singular right R-module is AWN-injective .Then $Y(R) \cap Z(R) = 0$.

Proof :

If $Y(R) \cap Z(R) \neq 0$, then there exists $0 \neq b \in Y(R) \cap Z(R)$ such that $b^2 = 0$. We claim that $RbR + r(b) = R$. Otherwise there exists a maximal essential right ideal M of R containing $RbR + r(b)$ .So R/M is AWN-injective, and $l_{R/M} r_R(b) = (R/M)b \oplus X_b$. $X_b \leq R/M$. Let $f : bR \to R/M$ be defined by $f(br) = r + M$. Note that f is a well defined R-homomorphism. Then $1 + M = f(b) = cb + M + x$, $c \in R$, $x \in X_b$, $1 - cb + M = x \in R/M \cap X_b = 0$, $1 - cb \in M$ .Since $cb \in RbR \subseteq M$, $1 \in M$, which is a contradiction . Therefore $1 = x + y$, $x \in RbR$, $y \in r(b)$, and so $b = bx$. Since $RbR \subseteq Z(R)$, $x \in Z(R)$. Thus $l(1-x) = 0$ and so $b = 0$, which is a contradiction . This show that $Y(R) \cap Z(R) = 0$.

Theorem 3.2:

R is a reduced ring if and only if R is a strongly right MC2 ring whose simple singular right R-modules are AWN-injective .

Proof :

The necessity is evident .

Conversely: Let $a^2 = 0$. Suppose that $a \neq 0$.Then there exists a maximal right ideal M of R containing $r(a)$. First observe that M is an essential right ideal of R.If not ,then $M = r(e)$ for some $e \in ME_r$ (the set of all minimal idempotents elements of R).Since R is strongly right MC2 ring , R is a strongly min-right semi central ring ,to so we obtain e is central in R. using $a \in r(a)$,we get $ae = ea = 0$ .Hence $e \in r(a) \subseteq M = r(e)$ .Which is a contradiction .Therefore M must be an essential right ideal of R .Thus R/M is AWN-injective ,and there exists a positive integer $n \geq 1$ such that $a^n \neq 0$ and $l_{R/M} r_R(a^n) = (R/M)a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$ .Since $a^2 = 0$,then $n = 1$,and therefore $l_{R/M} r_R(a) = (R/M)a \oplus X_a$ .Let $f : aR \to R/M$ defined by $f(ar) = r + M$ .Note that f is a well-defined R-homomorphism . Since R/M is AWN-injective ,there exists $c \in R$ such that $1 + M = f(a) = ca + M + x$, $x \in X_a$ (Lemma2.2). So $1 - ca + M = x \in R/M \cap X_a = 0$, Since $a^2 = 0$, $acaR \subseteq N^*(R)$ (the sum of all nil ideal) $\subseteq N(R)$ .Hence $ca \in N(R)$ and so $1 - ca \in U(R)$ (the set of all invertible elements), which implies that $M = R$, which is a contradiction . Therefore $a = 0$, and R is reduced .

Theorem 3.3:

Let R be a NCI ring .If R satisfies one of the following conditions ,then R is a reduced ring :

1- R is a right n-weakly regular .

2- Every simple right R-modules is AWN-injective .

3- R is right MC2 whose every simple singular right module is AWN-injective .

Proof :

If $N(R) \neq 0$, there exists $0 \neq I$ of R contained in $N(R)$.Clearly ,there exists $0 \neq b \in I$ such that $b^2 = 0$ and so there exists a maximal right ideal M of R containing $r(b)$.

If R is right n-weakly regular ,then $b = bc$ for some $c \in RbR$ .Since $RbR \subseteq I \subseteq N(R)$, there exists a positive integer $n \geq 1$ such that $c^n = 0$ Hence $b = bc = ccb = cccb = ...c^n b = 0$, which is a contradiction .

If R/M is AWN-injective, then $l_{R/M} r(b^n) = (R/M)b^n \oplus X_{b^n}$, $X_{b^n} \leq R/M$ .Since $b^2 = 0$,then $l_{R/M} r(b) = (R/M)b \oplus X_b$. Let $f : bR \to R/M$ be defined by $f(br) = r + M$. Note f is a well defined So $1 + M = f(b) = cb + M + x$, $c \in R, x \in X_b$, $1 - cb + M = x \in R/M \cap X_b = 0$,$1 - cb \in M$.

Since $cb \in I \subseteq N(R)$, $1 - cb \in U(R)$, which implies that $M = R$, a contradiction .

If M is not an essential right ideal of R, then $M = r(e)$ for some $e \in ME_r(R)$. Clearly $eb = 0$ .If $eRb \neq 0$, Then $eRbR = eR$ .But $eRbR \subseteq I \subseteq N(R)$,which is a contradiction , because $e \notin N(R)$ .So $eRb = 0$ .Therefore M is essential ,then R/M is AWN-injective and $l_{R/M} r(b) = (R/M)b \oplus X_b$ .Hence by the same method as in the proof of (2) , a contradiction . Therefore R is reduced .

A ring R is said to be NI if $N(R)$ forms an ideal of R. A ring R is said to be 2-prim if $N(R) = P(R)$, where $P(R)$ is the prime radical of R . Clearly , every 2-prime ring is NI [9] .

Theorem 3.4 :

Let R a right MC2 ring whose every Simple singular right R-module is AWN-injective, then the following conditions are equivalent :

1- R is reduced ring .

2- R is 2-prime ring .

3- R is NI ring .

Proof :

$1 \rightarrow 2 \rightarrow 3$ are obviously .

$(3) \rightarrow (1)$ Let $a^2 = 0$. Suppose $a \neq 0$. Then there exists a maximal right ideal M of R containing $r(a)$ .If M is not essential in R, then $M = r(e)$, where $e^2 = e \in R$ is a right minimal element . Hence $ea = 0$ because $a \in r(a)$ .If $eRa \neq 0$, then $eRaR = eR$. Since R is NI ring, then $N(R)$ is an ideal of R ,So $eRaR \in N(R)$ because $a \in N(R)$. Thus $e \in N(R)$,which is a contradiction .This show that $eRa = 0$. Hence $a\mathrm{Re} = 0$ because R is right MC2 . Thus $e \in r(a) \subseteq r(e)$ which is also a contradiction . This implies that M is essential in R ,then R/M is AWN-injective .by hypothesis .So $l_{R/M}r(a) = (R/M)a \oplus X_a$, $X_a \leq R/M$ ($a^2 = 0$,then n=1) .Let $f : aR \rightarrow R/M$ be defined by $f(ar) = r + M$. Note that f is well defined R-homomorphism . Then $1 + M = f(a) = ca + M + x$, $c \in R$, $x \in X_a$, $1 - ca + M = x \in R/M \cap X_a = 0$, $1 - ca \in M$. Since $ca \in N(R)$, $1 - ca$ is invertible ,So $M = R$,which is a contradiction . This show that $a = 0$ and so R is reduced .

Call a ring R right GMC2 for any $a \in R$, any right minimal idempotent $e \in R$, $eRa = 0$ implies $a\mathrm{Re} = 0$. Clearly, a right GMC2 ring is right MC2 . [12]

Lemma 3.5 [12]:

Let R be a right GMC2 ring and if $a \in R$ is not a right weakly regular element, then every maximal right ideal M of R containing $RaR + r(a)$ must be essential in R .

Proposition 3.6 :

Let R be a right GMC2 ring and if every simple singular right R-module is AWN-injective ,then for any $0 \neq a \in N(R)$ there exists a positive integer n such that $a^n \neq 0$ and $RaR + r(a^n) = R$.

Proof :

Assume that $a^n \neq 0$, $a^{n+1} = 0$. If a^n is a right weakly regular element ,then we are done. otherwise ,by Lemma(3.5), there exists a maximal essential right ideal containing $Ra^n R + r(a^n)$. Thus R/M is AWN-injective and $l_{R/M} r(a^n) = (R/M)a^n \oplus X_{a^n}$, $X_{a^n} \leq R/M$ .Let $f : a^n R \to R/M$ be defined by $f(a^n r) = r + M$ .Note that f is a well defined R-homomorphism. Then $1 + M = f(a^n) = da^n + M + x$, $d \in R$, $x \in X_{a^n}$, $1 - da^n + M = x \in R/M \cap X_{a^n} = 0$,$1 - da^n \in M$. Since $da^n \in Ra^n R \subseteq M$, $1 \in M$, which is a contradiction . Hence $R = Ra^n R + r(a^n) = RaR + r(a^n)$.

From Theorem (3.4) and proposition (3.6) we get :

Corollary 3.7:

Let R be a right GMC2 , NI ring ,whose every simple singular right R-module is AWN-injective .Then R is weakly regular ring .

Theorem3.8 :

If R is strongly right MC2 , then the following statements are equivalent :

Every right R-module is WN-injective .

Every right R-module is AWN-injective .

Every simple right R-module is AWN-injective .

Every simple singular right R-module is AWN-injective .

R is reduced.

R is n-regular .

Proof :

Obviously (1)$\Rightarrow$(2) $\Rightarrow$(3) $\Rightarrow$(4) , (5) $\Rightarrow$(6).And by [Theorem 3.2] , (4) implies (5) . (6) $\Rightarrow$(1) Lemma (2.1).

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