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ON Λ-GENERALIZED CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the concepts of Λ-generalized fuzzy closed sets(briefly, Λgf-closed sets), Λ-gf-closed sets and gf-Λ-closed sets in fuzzy topological spaces. Also we study some properties and characterizations of Λ-generalized fuzzy closed sets.

1. Introduction

In 1986, Maki [11] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (=saturated set), i.e the intersection of all open supersets of A. Arenas et al. [1] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. A subset A of a topological space (X, τ) is called λ-closed [1] if A =L∩D, where L is a Λ-set and D is a closed set. The intersection of all λ-closed sets containing a subset A of X is called the λ-closure of A and is denoted by $cl_\lambda(A)$ [5]. The complement of a λ-closed set is called λ-open. Ganster and Reilly [7] introduced the notion of locally closed sets using open sets and closed sets. In 1970, Levine [10] introduced the notion of generalized closed sets(briefly, g-closed sets) in topological spaces as a generalization of closed sets. Since

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then, many concepts related to generalized closed sets were defined and investigated. Caldas et al. [4] introduced new classes of sets called Λ_g-closed sets and Λ_g-open sets in topological spaces. They also established several properties of such sets. It is proved that Λ_g-closed sets and Λ_g-open sets are weaker forms of closed sets and open sets, respectively and stronger forms of g-closed sets and g-open sets, respectively.

Since the generalization of the usual notion of a set into a fuzzy set by Zadeh in his classic paper [17] of 1965, many abstract structures were generalized using fuzzy sets. Fuzzy topological spaces were introduced by Chang [6]. Fuzzy continuous functions and fuzzy closed functions were introduced by Chang in [6]. Recently Balasubramanian and Sundaram [3] introduced and studied the concepts of generalized fuzzy closed sets and fuzzy $T_{1/2}$-spaces in fuzzy topological spaces. Moreover, they studied the generalizations of fuzzy continuous functions.

In the present paper, we introduce the concepts of Λ-generalized fuzzy closed sets (briefly, Λgf-closed sets), Λ-gf-closed sets and gf-Λ-closed sets in fuzzy topological spaces. Further, we study some properties and characterizations of Λ-generalized fuzzy closed sets. Suitable Examples are given at proper places to substantiate the results. In topological spaces, the symbols such as $\subseteq$, $\cap$ and $\cup$ are used. Correspondingly, $\leq$, $\wedge$ and $\vee$ symbols are used in fuzzy topological spaces.

2. Preliminaries

A map from a nonempty set X into the closed unit interval I = [0, 1] is a fuzzy subset of X. The constant fuzzy sets taking the values 0 and 1 on X are denoted by 0_X and 1_X respectively. The family of all fuzzy sets of X is denoted by I^X. Usually the fuzzy sets will be denoted by Greek letters such as μ, ρ, ν, λ, α, β, ... or English alphabets such as A, B, C, ...

Definition 2.1. [6] *A family τ of fuzzy sets on X is called a fuzzy topology for X if (1) 0_X, $1_X \in \tau$, (2) $\mu \wedge \rho \in \tau$ whenever μ, $\rho \in \tau$ and (3) $\vee\{\mu_i : i \in \Delta\} \in \tau$ whenever each $\mu_i \in \tau (i \in \Delta)$.*

Moreover, the pair (X, τ) is called a fuzzy topological space.

Every member of τ is called a fuzzy open set.

The complement of a fuzzy open set is called a fuzzy closed set.

The complement of a fuzzy set λ of X is 1$-\lambda$(or λ^1).

Definition 2.2. *For a fuzzy set λ of (X, τ), the closure cl(λ) and the interior int(λ) of λ are defined in* [2] *respectively, as*

$cl(\lambda) = \wedge\{\nu : \nu \geq \lambda,\ \nu^1 \in \tau\}$ *and*

$int(\lambda) = \vee\{\nu : \nu \leq \lambda,\ \nu \in \tau\}$.

Definition 2.3. [2] *Let (X, τ) be a fuzzy topological space. A fuzzy set μ of X is called*

(1) *fuzzy regular open if μ=int(cl(μ));*

(2) *fuzzy regular closed if μ= cl(int(μ));*

It is easily seen that a fuzzy set μ is fuzzy regular open if and only if μ^1 is fuzzy regular closed.

Definition 2.4. [3] *A fuzzy set μ of a fuzzy topological space (X, τ) is called generalized fuzzy closed (briefly, gf-closed) if cl(μ)$\leq\lambda$ whenever $\mu\leq\lambda$ and $\lambda\in\tau$.*

Definition 2.5. [14] *A fuzzy set μ of a fuzzy topological space (X, τ) is called a fuzzy LC set if $\mu = \alpha \wedge \beta$ where α is a fuzzy open and β is a fuzzy closed.*

Definition 2.6. [3] *A fuzzy topological space (X, τ) is called fuzzy $T_{1/2}$ space if every gf-closed set is fuzzy closed.*

Definition 2.7. *Let μ be a fuzzy set of a fuzzy topological space (X, τ). Then μ is said to be*

(1) *fuzzy semiopen if and only if $\mu \leq$ cl(int(μ))* [2]*;*

(2) *fuzzy semiclosed if and only if μ^1 is a fuzzy semiopen set of X* [2]*;*

(3) *fuzzy preopen if and only if $\mu \leq$ int(cl(μ))* [15]*;*

(4) *fuzzy preclosed if and only if μ^1 is a fuzzy preopen set of X* [15].

Definition 2.8. *Let μ be a fuzzy set of a fuzzy topological space (X, τ). Then*

(1) *$pint(\mu) = \vee \{\lambda \mid \lambda \leq \mu, \lambda$ is a fuzzy preopen set of $X\}$, is called the fuzzy preinterior of μ* [8];
(2) *$pcl(\mu) = \wedge \{\lambda \mid \lambda \geq \mu, \lambda$ is a fuzzy preclosed set of $X\}$, is called the fuzzy preclosure of μ* [8];
(3) *$sint(\mu) = \vee \{\lambda \mid \lambda \leq \mu, \lambda$ is a fuzzy semiopen set of $X\}$, is called the fuzzy semiinterior of μ* [16];
(4) *$scl(\mu) = \wedge \{\lambda \mid \lambda \geq \mu, \lambda$ is a fuzzy semiclosed set of $X\}$, is called the fuzzy semiclosure of μ* [16].

Theorem 2.9. *Let μ be a fuzzy set in a fuzzy topological space (X, τ). Then*

(1) $pint(\mu) \leq \mu \wedge int(cl(\mu))$ [8];
(2) $pcl(\mu) \geq \mu \vee cl(int(\mu))$ [8];
(3) $sint(\mu) = \mu \wedge cl(int(\mu))$ [9];
(4) $scl(\mu) = \mu \vee int(cl(\mu))$ [9].

The following Lemma and two definitions are introduced in [6].

Definition 2.10. *Let $f : X \rightarrow Y$ be a function from a set X into a set Y, μ be a fuzzy subset in X and ρ be a fuzzy subset in Y. Then the Zadeh's functions $f(\mu)$ and $f^{-1}(\rho)$ are defined by*

(1) *$f(\mu)$ is a fuzzy subset of Y where*
$$f(\mu)=\begin{cases} \sup_{z \in f^{-1}(y)} \mu(z) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$
for each $y \in Y$.
(2) *$f^{-1}(\rho)$ is a fuzzy subset of X where $f^{-1}(\rho)(x)=\rho(f(x))$ for each $x \in X$.*
(3) $(f^{-1}(\rho))^1 = f^{-1}(\rho^1)$.

Lemma 2.11. *Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then for fuzzy sets μ and ρ of X and Y respectively, the following statements hold:*

(1) $ff^{-1}(\rho)\leq\rho$;
(2) $f^{-1}f(\mu)\geq\mu$;
(3) $f(\mu^1)\geq(f(\mu))^1$;
(4) $f^{-1}(\rho^1)=(f^{-1}(\rho))^1$;
(5) *if* f *is injective, then* $f^{-1}(f(\mu)) = \mu$;
(6) *if* f *is surjective, then* $ff^{-1}(\rho) = \rho$;
(7) *if* f *is bijective, then* $f(\mu^1) = (f(\mu))^1$.

Definition 2.12. *A function* $f : (X, \tau)\rightarrow(Y, \sigma)$ *is called*

(1) *fuzzy continuous if the inverse image of each fuzzy open set of* Y *is fuzzy open in* X,
(2) *fuzzy closed if the image of each fuzzy closed set of* X *is fuzzy closed in* Y.

Definition 2.13. [13] *Let* (X, τ) *be a fuzzy topological space. A fuzzy point* x_α $(0<\alpha\leq 1)$ *is a fuzzy set of* X *defined as follows:*

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.14. ([12], *Definition 1.2.4) Let* (X, τ) *be a fuzzy topological space. The fuzzy point* x_t *in* X *is said to be contained in a fuzzy set* μ *or to belong to* μ, *denoted by* $x_t \in \mu$, *if and only if* $t\leq\mu(x)$, *for each* $x \in X$. *Evidently every fuzzy set* μ *can be expressed as the union of all the fuzzy points which belong to* μ.

Definition 2.15. ([12], *Definition 8.2.3) A fuzzy topological space* (X, τ) *is called a* FT_1 *space if and only if every fuzzy point is a fuzzy closed set.*

Definition 2.16. [13] *Let* (X, τ) *be a fuzzy topological space. A fuzzy set* μ *is quasi-coincident with a fuzzy set* ν, *denoted by* $\mu q \nu$, *if there exists* $x\in X$ *such that* $\mu(x) + \nu(x)>1$. *If* μ *is not quasi-coincident with* ν, *then we write* $\mu\bar{q}\nu$. *It is known that* $\mu\leq\nu$ *if and only if* $\mu\bar{q}(1-\nu)$.

Definition 2.17. [13] *Let (X, τ) be a fuzzy topological space. A fuzzy point x_α in X is quasi-coincident with a fuzzy set ν, denoted by $x_\alpha q\nu$, if and only if $\alpha + \nu(x) > 1$. If x_α is not quasi-coincident with ν, then we write $x_\alpha \overline{q}\nu$.*

Lemma 2.18. [2] *For a fuzzy set λ of a fuzzy topological space (X, τ), the following hold:*

(1) $(int(\lambda))^1 = cl(\lambda^1)$;
(2) $(cl(\lambda))^1 = int(\lambda^1)$.

3. Λ-generalized fuzzy closed sets

Definition 3.1. *A fuzzy set μ of a fuzzy topological space (X, τ) is called a Λ-fuzzy set if $\mu = \hat{\mu}$, where $\hat{\mu} = \wedge\{\lambda : \mu\leq\lambda, \lambda\in\tau\}$.*

Remark 3.2. *For a fuzzy set μ of a fuzzy topological space (X, τ), the following properties hold:*

(1) $\mu\leq\hat{\mu}$.
(2) *If $\mu\in\tau$, then $\hat{\mu} = \mu$ and hence is a Λ-fuzzy set.*
(3) *If $\hat{\mu} = \mu$, then the following Example 3.3 shows that μ need not be fuzzy open.*
(4) *If $\mu\leq\sigma$, then $\hat{\mu}\leq\hat{\sigma}$.*
(5) *If μ is any fuzzy subset of X, then $\hat{\hat{\mu}} = \hat{\mu}$.*

Proof. (1), (2), (3), (4) Obvious.
(5) If $\rho\in\tau$ then $\mu\leq\rho\Leftrightarrow\hat{\mu}\leq\rho$ by the definition of $\hat{\mu}$ and (1). Hence $\hat{\hat{\mu}} = \hat{\mu}$ and $\hat{\mu}$ is a Λ-fuzzy set.

Example 3.3. *Let X be a nonempty set. Define $C_a : X\to[0, 1]$ such that $C_a(x) = a$ for all $x\in X$ and $a\in[0, 1]$. Then $\tau = \{C_0, C_1, C_a : 3/10 < a \leq 4/10\}$ is a fuzzy topology on X and (X, τ) is a fuzzy topological space. Now $\hat{C}_{3/10} = \wedge\{\rho: \rho\in\tau$ and $C_{3/10}\leq\rho\} = C_b$ where $b = \wedge\{a \mid 3/10 < a \leq 4/10\}$ and hence $b = 3/10$ and $\hat{C}_{3/10} = C_{3/10}$. Thus $C_{3/10}$ is a Λ-fuzzy set but not fuzzy open in X.*

Proposition 3.4. *In a FT_1 space (X, τ), every fuzzy subset of X is a Λ-fuzzy set.*

Proof. Let μ be any fuzzy subset of X such that $x_t\overline{q}\mu$. Then $\mu(x)\leq 1-t$. Since (X, τ) is a FT_1-space, x_t is a fuzzy closed set of X. Hence x_t^1 is fuzzy open containing μ. By definition of $\hat{\mu}$, $\hat{\mu}\leq x_t^1$ and therefore $x_t\overline{q}\hat{\mu}$. Thus $\mu(x)\leq 1-t \Rightarrow \hat{\mu}(x)\leq 1-t$ and hence $\hat{\mu}(x)\leq\mu(x)$. Then $\hat{\mu}=\mu$ and hence μ is a Λ-fuzzy set.

Definition 3.5. *A fuzzy set μ of a fuzzy topological space (X, τ) is said to be λ-fuzzy closed (briefly, λf-closed) if μ can be put in the form $\mu=\alpha\wedge\beta$ where α is a Λ-fuzzy set and β is fuzzy closed in X.*

The complement of a λf-closed set is λf-open. The collection of all λf-open sets in (X, τ) is denoted by $\lambda FO(X)$.

Lemma 3.6. *In a fuzzy topological space (X, τ), the following properties hold:*

1. *If μ_i is λf-closed for each $i\in\Delta$, then $\wedge_{i\in\Delta}\mu_i$ is λf-closed.*
2. *If μ_i is λf-open for each $i\in\Delta$, then $\vee_{i\in\Delta}\mu_i$ is λf-open.*
3. *Intersection of two λf-open sets is not necessarily λf-open.*

Proof. (1) Since μ_i is λf-closed, $\mu_i=\alpha_i\wedge\beta_i$ where α_i is a Λ-fuzzy set and β_i is fuzzy closed for each i. Therefore $\wedge_{i\in\Delta}\mu_i=\wedge_{i\in\Delta}(\alpha_i\wedge\beta_i)=[\wedge_{i\in\Delta}\alpha_i]\wedge[\wedge_{i\in\Delta}\beta_i]$.
Now $\wedge_{i\in\Delta}\alpha_i\leq\alpha_i$ for each i and by (4) of Remark 3.2.
We have $[\wedge_{i\in\Delta}\alpha_i]^{\wedge}\leq\hat{\alpha}_i$ for each i since each α_i is a Λ-fuzzy set.
Hence $[\wedge_{i\in\Delta}\alpha_i]^{\wedge}\leq\wedge_{i\in\Delta}\alpha_i$ and thus $\wedge_{i\in\Delta}\alpha_i$ is a Λ-fuzzy set.
Since β_i is fuzzy closed for each i, $\wedge_{i\in\Delta}\beta_i$ is fuzzy closed and hence $\wedge_{i\in\Delta}\mu_i$ is λf-closed.

(2) Taking complements the proof follows from (1).

(3) Let X = {a, b} and A_1 : X → [0, 1] be defined as $A_1(a)=0.4$ and $A_1(b)=0.6$. Then $\tau=\{0_X, 1_X, A_1\}$ is a fuzzy topology on X with A_1 = {(a, 0.4), (b, 0.6)}. In (X, τ), A_1 and $A_2={A_1}^1$ are λf-open subsets by (4) of Remark 3.7. $(A_1\wedge A_2)^1$ is not λf-closed, since the only Λ-fuzzy sets of X are 0_X, A_1 and

1_X and the only fuzzy closed sets of X are 0_X, A_2, 1_X. Thus $A_1 \wedge A_2$ is not λf-open.

Remark 3.7. *The following statements are true for any fuzzy topological space.*

1. *Every Λ-fuzzy set of X is λf-closed in X.*
2. *Every fuzzy closed set of X is λf-closed in X.*
3. *Every fuzzy LC set of X is λf-closed in X.*
4. *Every fuzzy open set of X is both λf-open and λf-closed.*

Proof. (1) Let μ be a Λ-fuzzy set. Then $\mu = \mu \wedge I_X$ where I_X is fuzzy closed of X. Hence μ is λf-closed in X.

(2) Let μ be a fuzzy closed set of X. Then $\mu = I_X \wedge \mu$ where I_X is a Λ-fuzzy set of X. Hence μ is λf-closed in X.

(3) Let μ be a fuzzy LC set. Then $\mu = \alpha \wedge \beta$ where α is a fuzzy open set and β is a fuzzy closed set of X. By (2) of Remark 3.2, α is a Λ-fuzzy set of X and hence μ is λf-closed in X.

(4) Let μ be a fuzzy open of X. By (2) of Remark 3.2 and (1) of Remark 3.7, μ is λf-closed. Again by (2) of Remark 3.7, μ is λf-open. Thus μ is both λf-closed and λf-open in X.

The converse of each statement in Remark 3.7 is not true can be shown by the following Example.

Example 3.8. (1) *In Example 3.3, $C_{6/10}$ is fuzzy closed since $C_{4/10} \in \tau$. By (2) of Remark 3.7, $C_{6/10}$ is λf-closed. But $\hat{C}_{6/10} = C_1 \neq C_{6/10}$ and thus it is not a Λ-fuzzy set.*

(2) *In Example 3.3, $C_{3/10}$ is a Λ-fuzzy set since $\hat{C}_{3/10} = C_{3/10}$ and hence by (1) of Remark 3.7, $C_{3/10}$ is λf-closed but not fuzzy closed in X.*

(3) *In Example 3.3, $C_{3/10}$ is λf-closed by (1) of Remark 3.7. But $C_{3/10}$ is not a fuzzy LC set. If $C_{3/10} = \alpha \wedge \beta$ where α is fuzzy open and β is fuzzy closed in X, then $C_{3/10} = C_b \wedge C_d$. Since C_b is fuzzy open, $b > 3/10$ and C_d is fuzzy closed*

implies d≥6/10. Thus 3/10 = min{b,d} which is a contradiction, proving that $C_{3/10}$ is not a fuzzy LC set of X.

(4) *In Example 3.3, $C_{6/10}$ is λf-closed by (2) of Remark 3.7. Since $C_{4/10}$ is fuzzy open, by (4) of Remark 3.7, $C_{4/10}$ is λf-closed and hence $C_{6/10}$ is λf-open. Thus $C_{6/10}$ is both λf-closed and λf-open but not fuzzy open.*

Remark 3.9. *From (1) and (2) of Example 3.8, it is easy to see that Λ-fuzzyness and fuzzy closedness are independent.*

Lemma 3.10. *For a fuzzy set μ of a fuzzy topological space (X, τ), the following conditions are equivalent.*

(1) *μ is λf-closed.*
(2) *$\mu = L \wedge cl(\mu)$ where L is a Λ-fuzzy set.*
(3) *$\mu = \hat{\mu} \wedge cl(\mu)$.*

Proof. (1)⇒(2). Obvious since cl(μ) is fuzzy closed containing μ.
(2)⇒(3). Obvious since $\hat{\mu}$ is a Λ-fuzzy set, by (5) of Remark 3.2.
(3)⇒(1). Follows since $\hat{\mu}$ is a Λ-fuzzy set.

Lemma 3.11. *A fuzzy set μ of a fuzzy topological space (X, τ) is gf-closed if and only if $cl(\mu) \leq \hat{\mu}$.*

Proof. Let μ be gf-closed in X. Then $cl(\mu) \leq \lambda$, whenever $\mu \leq \lambda$ for any $\lambda \in \tau$. Thus $cl(\mu) \leq \wedge\{\lambda : \mu \leq \lambda \text{ and } \lambda \in \tau\} = \hat{\mu}$.

Conversely. Let $\mu \leq \lambda$ and $\lambda \in \tau$. By the definition of $\hat{\mu}$, $\hat{\mu} \leq \lambda$. Then $cl(\mu) \leq \hat{\mu} \leq \lambda$. Thus $cl(\mu) \leq \lambda$ whenever $\mu \leq \lambda$ and $\lambda \in \tau$, which proves that μ is gf-closed.

Theorem 3.12. *For a fuzzy set μ of a fuzzy topological space (X, τ), the following conditions are equivalent.*

(1) *μ is fuzzy closed.*
(2) *μ is gf-closed and a fuzzy LC set.*
(3) *μ is gf-closed and λf-closed.*

Proof. (1)⇒(2). Since every fuzzy closed set is gf-closed and also a fuzzy LC set, the proof follows immediately.

(2)⇒(3). By (3) of Remark 3.7, every fuzzy LC set is λf-closed and hence the proof.

(3)⇒(1). Since μ is gf-closed, then by Lemma 3.11, cl(μ)$\leq\hat{\mu}$. But μ is λf-closed, hence by Lemma 3.10, $\mu = \hat{\mu}\wedge$cl(μ). Therfore $\mu =$ cl(μ) and thus μ is fuzzy closed.

Definition 3.13. *For a fuzzy set μ of a fuzzy topological space (X, τ), $cl_\lambda(\mu)$ is defined as the intersection of all λf-closed sets containing μ and is called the λf-closure of μ.*

Remark 3.14. *The following properties hold in any fuzzy topological space:*

1. *For a fuzzy set μ, $\mu\leq cl_\lambda(\mu)$.*
2. *$cl_\lambda(\mu)$ is λf-closed for a fuzzy set μ.*
3. *If μ is λf-closed, then $\mu = cl_\lambda(\mu)$.*
4. *If $\mu\leq\sigma$, then $cl_\lambda(\mu)\leq cl_\lambda(\sigma)$.*
5. *For a fuzzy set μ, $cl_\lambda(\mu)\leq cl(\mu)$.*

Definition 3.15. *A fuzzy set μ of a fuzzy topological space (X, τ) is called Λ-generalized fuzzy closed (briefly, Λgf-closed) (respectively, Λ-gf-closed, gf-Λ-closed) if cl(μ)$\leq\beta$ (respectively, $cl_\lambda(\mu)\leq\beta$, $cl_\lambda(\mu)\leq\beta$) whenever $\mu\leq\beta$ and β is λf-open (respectively, β is λf-open, β is fuzzy open).*

As a consequence of the above definition, we have the following Proposition.

Proposition 3.16. *For a fuzzy topological space (X, τ), then the following properties hold.*

1. *Every fuzzy closed set is Λgf-closed.*
2. *Every Λgf-closed set is gf-closed.*
3. *Every λf-closed set is Λ-gf-closed.*
4. *Every Λ-gf-closed set is gf-Λ-closed.*
5. *Every Λgf-closed set is Λ-gf-closed.*
6. *Every gf-closed set is gf-Λ-closed.*

Proof. (1) Let μ be a fuzzy closed set and λ be any λf-open set containing μ. Then cl(μ) = $\mu \leq \lambda$. Thus cl(μ)$\leq\lambda$ and hence μ is Λgf-closed.

(2) Since any fuzzy open set λ is λf-open by (2) of Remark 3.7, the proof follows.

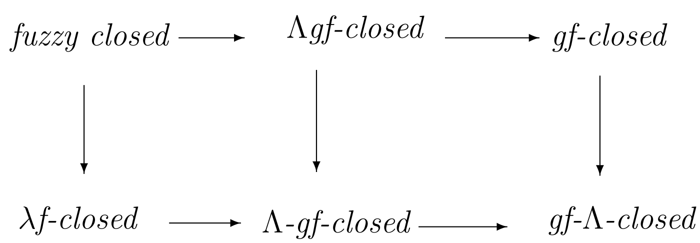
(3) Proof follows from (3) of Remark 3.14.

(4) Proof follows from (2) of Remark 3.7.

(5) Proof follows from (5) of Remark 3.14.

(6) Proof follows from (5) of Remark 3.14.

Remark 3.17. *From the above discussions, we have the following diagram:*



Remark 3.18. *From the single Example 3.3 it is seen that none of the above implications is reversible.*

The different types of fuzzy sets other than C_0 and C_1 in Example 3.3 are

fuzzy open sets = $\{C_a : 3/10 < a \leq 4/10\}$;

fuzzy closed sets = $\{C_a : 6/10 \leq a < 7/10\}$;

Λ-fuzzy sets = $\{C_a : 3/10 \leq a \leq 4/10\}$;

λf-closed sets = $\{C_a : 3/10 \leq a \leq 4/10,\ 6/10 \leq a < 7/10\}$;

λf-open sets = $\{C_a : 3/10 < a \leq 4/10,\ 6/10 \leq a \leq 7/10\}$;

gf-closed sets = $\{C_a : 4/10 < a \leq 1\}$;

Λ-gf-closed sets = $\{C_a : 0 \leq a < 7/10,\ 7/10 < a \leq 1\}$;

Λgf-closed sets = $\{C_a : 4/10 < a < 7/10,\ 7/10 < a \leq 1\}$;

and gf-Λ-closed sets = $\{C_a : 0 \leq a \leq 1\}$.

(1) *Λgf-closed$\nrightarrow$fuzzy closed.*

The λf-open sets containing $C_{5/10}$ are $\{C_a : 6/10 \leq a \leq 7/10\}$. But cl($C_{5/10}$) = $C_{6/10} \leq \{C_a : 6/10 \leq a \leq 7/10\}$. Thus cl($C_{5/10}$)$\leq\lambda$ whenever $C_{5/10}\leq\lambda$

and λ is λf-open, which proves that $C_{5/10}$ is Λgf-closed. But $C_{5/10}$ is not fuzzy closed.

(2) *gf-closed $\nrightarrow$ Λgf-closed.*

For $C_{7/10}$, C_1 is the only fuzzy open set containing $C_{7/10}$ and hence $C_{7/10}$ is gf-closed. Since $C_{7/10}$ is λf-open set containing $C_{7/10}$ we have $C_{7/10} \leq C_{7/10}$. But cl($C_{7/10}$) $= C_1 \nleq C_{7/10}$. Hence $C_{7/10}$ is not Λgf-closed. Thus $C_{7/10}$ is gf-closed but not Λgf-closed.

(3) *Λ-gf-closed$\nrightarrow$$\lambda$f-closed.*

$C_{5/10}$ is Λgf-closed and therefore Λ-gf-closed. But $C_{5/10}$ is not λf-closed.

(4) *gf-Λ-closed$\nrightarrow$$\Lambda$-gf-closed.*

By (2) $C_{7/10}$ is gf-closed and therefore gf-Λ-closed. Now $C_{7/10}$ is λf-open and $C_{7/10} \leq C_{7/10}$. But $cl_\lambda(C_{7/10}) = C_1 \nleq C_{7/10}$ and hence $C_{7/10}$ is not Λ-gf-closed.

(5) *λf-closed$\nrightarrow$ fuzzy closed.*

$C_{3/10}$ is a Λ-fuzzy set and therefore it is λf-closed, but $C_{3/10}$ is not fuzzy closed.

(6) *Λ-gf-closed$\nrightarrow$$\Lambda$gf-closed.*

By (5) $C_{3/10}$ is λf-closed and hence Λ-gf-closed. But $C_{4/10}$ is fuzzy open and hence λf-open with $C_{3/10} \leq C_{4/10}$ whereas cl($C_{3/10}$) $= C_{6/10} \nleq C_{4/10}$ which proves that $C_{3/10}$ is not Λgf-closed.

(7) *gf-Λ-closed$\nrightarrow$gf-closed.*

$cl_\lambda(C_{4/10}) = C_{4/10}$ and hence $cl_\lambda(C_{4/10}) \leq \lambda$ whenever $C_{4/10} \leq \lambda$ and λ is fuzzy open. Thus $C_{4/10}$ is gf-Λ-closed. But $C_{4/10}$ is fuzzy open and $C_{4/10} \leq C_{4/10}$ whereas cl($C_{4/10}$) $= C_{6/10} \nleq C_{4/10}$. This proves that $C_{4/10}$ is not gf-closed.

Remark 3.19. *The following concepts are independent. This fact can be seen from the Examples given.*

(1) *Λgf-closedness and λf-closedness.*

Example (1) : In Example 3.3, $C_{3/10}$ is λf-closed. But $C_{3/10}$ is not Λgf-closed by (6) of Remark 3.18.

Example (2) : In Example 3.3, $C_{5/10}$ *is Λgf-closed by (1) of Remark 3.18. But* $C_{5/10}$ *is not λf-closed.*

(2) *Λ-gf-closedness and gf-closedness.*

Example (1) : In Example 3.3, $C_{4/10}$ *is Λ-gf-closed since* $cl_\lambda(C_{4/10}) = C_{4/10}$ *by (7) of Remark 3.18. But it is not gf-closed by (7) of Remark 3.18.*

Example (2) : In Example 3.3, $C_{7/10}$ *is gf-closed by (2) of Remark 3.18. But* $C_{7/10}$ *is not Λ-gf-closed by (4) of Remark 3.18.*

(3) *λf-closedness and gf-closedness.*

Example (1) : In Example 3.3, $C_{3/10}$ *is λf-closed by (5) of Remark 3.18. But* $C_{3/10}$ *is not gf-closed for* $C_{4/10}$ *is fuzzy open such that* $C_{3/10} \leq C_{4/10}$ *whereas* $cl(C_{3/10}) = C_{6/10} \not\leq C_{4/10}$.

Example (2) : In Example 3.3, $C_{7/10}$ *is gf-closed by (2) of Remark 3.18. By (4) of Remark 3.18,* $C_{7/10}$ *is not Λ-gf-closed and hence not λf-closed.*

Remark 3.20. (1) *Decomposition of a fuzzy closed set in terms of λf-closedness and gf-closedness.*

By Theorem 3.12, a fuzzy set μ is fuzzy closed⇔μ is λf-closed and gf-closed. By (3) of Remark 3.19, λf-closedness and gf-closedness are independent.

(2) *Decomposition of a fuzzy closed set in terms of λf-closedness and Λgf-closedness.*

Let μ be fuzzy closed in (X, τ). By Proposition 3.16, μ is Λgf-closed. Also by (2) of Remark 3.7, μ is λf-closed. Hence μ is Λgf-closed and λf-closed.

Conversely. Let μ be Λgf-closed and λf-closed. By Proposition 3.16, μ is gf-closed and hence by Theorem 3.12, μ is fuzzy closed. Thus μ is fuzzy closed⇔μ is Λgf-closed and λf-closed. By (1) of Remark 3.19, Λgf-closedness and λf-closedness are independent.

Theorem 3.21. *The union of two Λgf-closed sets is Λgf-closed.*

Proof. Let A and B be any two Λgf-closed sets of a fuzzy topological space (X, τ). Let A∨B ≤ U, where U is λf-open. Then A≤U and B≤U. Since A and B are Λgf-closed, cl(A)≤U and cl(B)≤U. Hence cl(A∨B) = cl(A)∨cl(B)≤U. Thus A∨B is Λgf-closed in X.

Remark 3.22. *The intersection of two Λgf-closed sets need not be Λgf-closed as can be verified by the following Example.*

Example 3.23. *Let X={a, b} and A : X→[0, 1] be defined as A(a) = 0.2, A(b) = 0.2. Then (X, τ) is a fuzzy topological space with τ ={0_X, 1_X, A}. The Λ-fuzzy sets in X are 0_X, 1_X and A. The fuzzy closed sets are 0_X, 1_X and A^1. Hence the λf-closed sets are 0_X, 1_X, A and A^1. The λf-open sets are 0_X, 1_X, A and A^1 where A = {(a, 0.2), (b, 0.2)} and A^1 = {(a, 0.8), (b, 0.8)}. μ_1 = {(a, 0.2), (b, 1)} is Λgf-closed since 1_X is the only λf-open set in X containing μ_1. And μ_2 = {(a, 1), (b, 0.2)} is also Λgf-closed since 1_X is the only λf-open set in X containing μ_2. But $\mu_1 \wedge \mu_2$ = {(a, 0.2), (b, 0.2)} which is fuzzy open and hence λf-open, with {(a, 0.2), (b, 0.2)}≤{(a, 0.2), (b, 0.2)} whereas cl{(a, 0.2), (b, 0.2)} = {(a, 0.8), (b, 0.8)}≰{(a, 0.2), (b, 0.2)}. This verifies that $\mu_1 \wedge \mu_2$ is not Λgf-closed, inspite of μ_1 and μ_2 being Λgf-closed in X.*

Proposition 3.24. *If A is a Λgf-closed set of (X, τ) and A≤B≤cl(A), then B is a Λgf-closed set of (X, τ).*

Proof. Let B≤U where U is λf-open in X. Since A≤B, A≤U. But A is Λgf-closed set in X, then cl(A)≤U. Also, B≤cl(A), cl(B)≤cl(A)≤U. Therefore B is Λgf-closed in X.

Theorem 3.25. *If A is a λf-open and Λgf-closed set of (X, τ), then A is fuzzy closed in X.*

Proof. Since A is λf-open and Λgf-closed, then cl(A)≤A and hence A is fuzzy closed in X.

Theorem 3.26. *Let A be a Λgf-closed set in (X, τ).*

(1) *If A is fuzzy regular open, then scl(A) and pint(A) are Λgf-closed sets.*

(2) *If A is fuzzy regular closed, then pcl(A) and sint(A) are* Λ*gf-closed sets.*

Proof. (1) By the definitions, scl(A) $\geq$ A and pint(A) $\leq$ A. But A is fuzzy regular open, then A = int(cl(A)). Therefore A is fuzzy semiclosed and fuzzy preopen. Thus A is fuzzy semiclosed and hence A = scl(A). Similarly we obtain A = pint(A). Therefore scl(A) and pint(A) are Λgf-closed sets.
(2) By the definitions, pcl(A) $\geq$ A and sint(A) $\leq$ A. But A is fuzzy regular closed, then A = cl(int(A)). Therefore A is fuzzy preclosed and fuzzy semiopen. Thus A = pcl(A) and A = sint(A). Therefore pcl(A) and sint(A) are Λgf-closed sets.

Theorem 3.27. *Let* (X, τ) *be a fuzzy* $T_{1/2}$ *space. Then the following conditions are equivalent.*

(1) *A is fuzzy closed.*
(2) *A is* Λ*gf-closed.*
(3) *A is gf-closed.*

Proof. Obvious.

Definition 3.28. *A fuzzy set A in* (X, τ) *is said to be* Λ*gf-open in* (X, τ) *if and only if* A^1 *is* Λ*gf-closed in* (X, τ)*.*

It is evident that every fuzzy open set of (X, τ) *is* Λ*gf-open in* (X, τ) *but not conversely.*

In Example of Remark 3.18, $C_{5/10}$ *is* Λ*gf-open but not fuzzy open in X.*

Theorem 3.29. *The intersection of two* Λ*gf-open sets in* (X, τ) *is* Λ*gf-open.*

Proof. This is obvious by Theorem 3.21.

The following Example shows that arbitrary union of Λgf-closed sets is not necessarily Λgf-closed.

Example 3.30. *In Example 3.3,* Λ*gf-closed sets* = $\{C_a : 4/10 < a < 7/10$ *and* $7/10 < a \leq 1\}$ *by Remark 3.18.* C_a *is* Λ*gf-closed for each a such that* $4/10 < a < 7/10$

and $\vee C_a = C_b$ *where* $b = \vee\{a \mid 4/10 < a < 7/10\}$. *Hence* $\vee C_a = C_{7/10}$, *where* $4/10 < a < 7/10$, *which is not* Λ*gf-closed.*

Theorem 3.31. *A fuzzy set A is* Λ*gf-open in (X,* τ*) if and only if F*$\leq$*int(A) whenever F is* λ*f-closed in (X,* τ*) and F*$\leq$*A.*

Proof. Sufficiency. We first prove that A^1 is Λgf-closed. Let $A^1 \leq G$, where G is λf-open. Hence $G^1 \leq A$ and G^1 is λf-closed. Then by the assumption $G^1 \leq \text{int}(A)$ which implies that $(\text{int}(A))^1 \leq G$, so $\text{cl}(A^1) \leq G$. Hence A^1 is Λgf-closed i.e., A is Λgf-open.

Conversely. Let A be Λgf-open. Then A^1 is Λgf-closed and let F be a λf-closed set contained in A. Then $A^1 \leq F^1$. But A^1 is Λgf-closed, then $\text{cl}(A^1) \leq F^1$. This implies that $F \leq (\text{cl}(A^1))^1 = \text{int}(A)$. Thus $F \leq \text{int}(A)$.

Proposition 3.32. *If int(A)*$\leq$*B*$\leq$*A and A is* Λ*gf-open in (X,* τ*), then B is* Λ*gf-open in (X,* τ*).*

Proof. Suppose $\text{int}(A) \leq B \leq A$ and A is Λgf-open in (X, τ). Then $A^1 \leq B^1 \leq \text{cl}(A^1)$ and A^1 is Λgf-closed. By Proposition 3.24, B^1 is Λgf-closed in (X, τ) and hence B is Λgf-open in (X, τ).

4. Fuzzy functions

We begin with the following notions:

Definition 4.1. *A fuzzy function f : (X,* τ*)*$\rightarrow$*(Y,* σ*) is said to be*

1. λ*f-irresolute if* $f^{-1}(V)$ *is* λ*f-open in X for every* λ*f-open set V of Y.*
2. λ*f-closed if f(F) is* λ*f-closed in Y for every* λ*f-closed set F of X.*

Theorem 4.2. *Let f : (X,* τ*)*$\rightarrow$*(Y,* σ*) be a* λ*f-irresolute fuzzy closed function. If A is* Λ*gf-closed in X, then f(A) is* Λ*gf-closed in Y.*

Proof. Let A be a Λgf-closed set of X and V be a λf-open set of Y containing f(A). But f is λf-irresolute, then $f^{-1}(V)$ is λf-open in X and $A \leq f^{-1}(V)$. Since A is Λgf-closed, hence $\text{cl}(A) \leq f^{-1}(V)$ and $f(A) \leq f(\text{cl}(A)) \leq V$. Since f is fuzzy closed, we obtain $\text{cl}(f(A)) \leq V$. This shows that f(A) is Λgf-closed in Y.

Lemma 4.3. *A function f : (X, τ)→(Y, σ) is λf-closed if and only if for each fuzzy subset B of Y and each U∈λFO(X) containing f^{-1}(B), there exists V∈λFO(Y) such that B≤V and f^{-1}(V)≤U.*

Proof. Necessity. Suppose that f is a λf-closed function. Let B≤Y and U∈λFO(X) containing f^{-1}(B). Put $V = (f(U^1))^1$. Then we obtain V∈λFO(Y), B≤V and f^{-1}(V)≤U.

Sufficiency. Let F be any λf-closed set of (X, τ). Set f(F) = B, then F≤ f^{-1}(B) and $f^{-1}(B^1) \leq F^1$∈λFO(X). By hypothesis, there exists V∈λFO(Y) such that $B^1 \leq V$ and $f^{-1}(V) \leq F^1$. Therefore we obtain $V^1 \leq B = f(F) \leq V^1$. Hence f(F) = V^1 and f(F) is λf-closed in (Y, σ). Therefore, f is λf-closed.

Theorem 4.4. *Let f : (X, τ)→(Y, σ) be a fuzzy continuous λf-closed function. If B is a Λgf-closed set of (Y, σ), then f^{-1}(B) is Λgf-closed in (X, τ).*

Proof. Let B be a Λgf-closed in (Y, σ) and U be a λf-open set of (X, τ) containing f^{-1}(B). Since f is λf-closed, then by Lemma 4.3 there exists a λf-open set V of (Y, σ) such that B≤V and f^{-1}(V)≤U. But B is Λgf-closed in (Y, σ), then cl(B)≤V and hence $f^{-1}(B) \leq f^{-1}(cl(B)) \leq f^{-1}(V) \leq U$. Since f is fuzzy continuous, f^{-1}(cl(B)) is fuzzy closed and hence cl(f^{-1}(B))≤U. This shows that f^{-1}(B) is Λgf-closed in (X, τ).

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