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MONOTONIC ANALYSIS: SOME RESULTS OF INCREASING AND POSITIVELY HOMOGENEOUS FUNCTIONS

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Abstract. The theory of increasing and positively homogeneous (IPH) functions defined on a convex cone in a topological vector space X, is well developed. In this article, we present necessary and sufficient conditions for the minimum of the difference of strictly IPH functions defined on X. We study convergence of sequences of increasing positively homogeneous (IPH) functions defined on X.

1. Introduction

Recently many authors have discussed the theoretical development of optimality conditions for certain classes of global optimization problems (see [1,2]). One of the most important global optimization problems is to minimize a DC function (difference of two convex functions) that is

$$minimize\, h(x)\; subject\, to\, x \in X,$$

where $h(x) = q(x) - p(x)$ and p, q are convex functions. In a general case, DC functions can be replaced by DAC functions (difference of two abstract convex functions)[5]. In this paper, we replace p and q by increasing positively homogeneous (IPH) functions and we present a necessary and sufficient condition for the global minimum of h. Then we consider four different types of convergence for sequences of

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IPH functions defined on X. In particular pointwise convergence and epi-convergence. We shall use the following notations:

$$\mathbb{R} = (-\infty, +\infty); \qquad \bar{\mathbb{R}} = [-\infty, +\infty]; \qquad \bar{\mathbb{R}}_+ = [0, +\infty].$$

2. Main Results

Let X be a topological vector space. A set$K \subseteq X$ is called conic, if $\lambda k \subseteq K$ for all $\lambda > 0$. We assume that X is equipped with a closed convex pointed cone K (the latter means that $K \cap -K = 0$). The increasing property of our functions taken with respect to the ordering $\leq$ induced on X by K:

$$x \leq y \Longleftrightarrow y - x \in K$$

A function $p : X \longrightarrow \bar{\mathbb{R}}$ is called positively homogeneous if

$$p(\lambda x) = \lambda p(x).$$

for all $x \in X$ and $\lambda \geq 0$. The function p is called increasing if $x \geq y \rightarrow p(x) \geq p(y)$. We shall study increasing positively homogeneos (briefly IPH) functions definded on X. Denote the set of all such functions by $\mathcal{P}(X)$.

A function $p : X \longrightarrow \mathbb{R}_{+\infty}$ is called proper if $domf \neq \emptyset$, where dom f is defined by $domf = \{x \in X : f(x) < +\infty\}$.

Now, consider the function $l : X \times X \rightarrow \bar{\mathbb{R}}_+$ defined by:

$$l(x, y) := \max\{\lambda \geq 0 \ : \ \lambda y \leq x\}$$

(with the convention $. \max \mathbb{R} := +\infty$, $max\, \emptyset := 0$)

In the sequel, for each $y \in X$, consider the coupling function $l_y : X \rightarrow \bar{\mathbb{R}}_+$ defined by $l_y(x) := l(x, y)$ for all $x \in X$, and set $L := \{l_y : y \in X\}$.

Let $X' = X \backslash (-K)$ and $L' = \{L_y : y \in X'\}$.

Theorem 2.1. *The mapping $\psi : X' \to L'$ defined by $\psi(y) := l_y$ is a bijection from X' onto L', and*

$$y_1 \leq y_2 \Longleftrightarrow l_{y_2} \leq l_{y_1} \qquad y_1, y_2 \in X'$$

Proof. Since, by the definition of L', ψ is obviously onto. Thus we only have to prove that ψ is one-to-one. Assume that $y_1, y_2 \in X'$ are such that $l_{y_1} = l_{y_2}$. Thus $1 = l(y_1, y_1) = l(y_1, y_2)$. Hence, we get $y_2 \leq y_1$. By symmetry it follows that $y_2 \geq y_1$. Since K is pointed, we conclude that $y_2 = y_1$.

Assume now that $l_{y_2} \leq l_{y_1}$. Then either $y_2 = 0$, whence $l_{y_2} = +\infty = l_{y_1}$. So that $y_1 = y_2$, or $y_2 \neq 0$ and hence,

$$1 = l_{y_2}(y_2) \leq l_{y_1}(y_2) = \max\{\lambda \geq 0 \ : \ \lambda y_1 \leq y_2\};$$

Which implies that $y_1 \leq y_2$, the converse follows from definition of l . □

The lower polar function of $p : X \to \bar{\mathbb{R}}_+$ is the function $p^0 : L \to \bar{\mathbb{R}}_+$ defined by:

$$p^0(l_y) = \sup_{x \in X} \frac{l_y(x)}{p(x)} \qquad , \qquad l_y \in L.$$

Theorem 2.2. *([5]). Let $p : X \to \bar{\mathbb{R}}_+$ be a function. Then p is IPH if and only if*

$$p^0(l_y) = \frac{1}{p(y)} \qquad , \forall l_y \in \ L.$$

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Proposition 2.1. *([5]). Let $p : X \to \bar{\mathbb{R}}_+$ be an IPH function. Then p is IPH, if and only if*

$$supp(p, L) = \{l_y \in L \ : \ p(y) \geq 1\}.$$

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3. Necessary and Sufficient Conditions

In this section, we present necessary and sufficient conditions for the global minimum of the difference of strictly IPH functions. Recall that, $p : X \to \bar{\mathbb{R}}$ for a function, $x_0 \in X$ is a global minimizer of the function p if:

$$-\infty < p(x_0) \leq p(x), \qquad \forall x \in X.$$

First, consider the function $h := q - p$, where $p, q : X \to \bar{\mathbb{R}}$ are proper functions. Let $\eta := \inf_{x \in X} h(x)$. This implies that $p(x) \leq q(x) - \eta$, $\forall x \in X$. Let $\tilde{q}(x) := q(x) - \eta$. It is easy to see that $p(x) \leq \tilde{q}(x)$ for all $x \in X$ if and only if $supp(p, L) \subset supp(\tilde{q}, L)$, or equivalently, x_0 is a global minimizer of the function h if and only if

$$supp(p, L) \subset supp(\tilde{q}, L).$$

Now, consider a set U of functions defined on a set X. We assume that U is equipped with the natural (pointwise) order relation. Recall that a function f is called a maximal element of the set U, if $f \in U$ and $\bar{f} \in U$, $\bar{f}(x) \geq f(x)$ for all $x \in X \to \bar{f} = f$. We now concentrate on the support set of IPH functions and we obtain some results which will be used later.

Proposition 3.1. *Let $p : X \to \bar{\mathbb{R}}$ be an IPH function and let $l_y \in supp(p, L)$. Assume that l_y is a maximal element of $supp(p, L)$. Then $p(y) = 1$.*

Proof. Let $l_y \in supp(p, L)$ then by Proposition (2.1), we have $p(y) \geq 1$. Consider $l_{(y, \frac{y}{p(y)})} \in L$. Then, in view of the definition of l_y we conclude that

$$l_{(y, \frac{y}{p(y)})} = p(y).$$

Since $p(y) \geq 1$, it follows from Proposition (2.1) that

$$l_{(y, \frac{y}{p(y)})} \in supp(p, L).$$

Also, by using $p(y) \geq 1$ and the definition of l_y one has

$$l_y(x) \leq l_{\frac{y}{p(y)}}(x), \qquad \forall x \in X.$$

Since l_y is a maximal element of supp(p,L) , then we obtain

$$l_y(x) = l_{\frac{y}{p(y)}}(x), \qquad \forall x \in X. \tag{3.1}$$

Put $x := y$ in (1), we get $p(y) = 1$. □

The converse statement of Proposition (3.1) is not valid. Consider IPH function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x$ for all $x \in \mathbb{R}$. It follows from Proposition (2.1) that $l_1 \in supp(f, L)$ and $f(1) = 1$. But, l_1 is not maximal element of the support set of f .

Proposition 3.2. *Let $p : X \to \bar{\mathbb{R}}$ be a strictly IPH function and let $l_y \in supp(p, L)$. Then l_y is a maximal element of $supp(p, L)$ if and only if $p(y) = 1$.*

Proof. Due to Proposition (3.1) we only show that if $p(y) = 1$. Thus l_y is a maximal element for the support set of p. Consider $l_{y'} \in supp(p, L)$ such that $l_y(x) \leq l_{y'}(x)$ for all $x \in X$. We are going to show that $l_y = l_{y'}$. We have

$$1 = l_y(y) \leq l_{y'}(y).$$

Consider the element $\bar{y} = \frac{y}{p(y)}$. We have

$$1 = p(y) = l_y(\bar{y}) \leq l_{y'}(\bar{y}) \leq p(\bar{y}) = 1$$

Then, $l_{y'}(\bar{y}) = 1$. This, together with $p(y) = 1$ imply that $y' \leq y$. Now since p is strictly increasing and $y' \leq y$, we obtain $y = y'$, which completes the proof. □

Proposition 3.3. *Let $p : X \to \bar{\mathbb{R}}$ be a strictly IPH function. Then for each $l_y \in supp(p, L)$ there exists a maximal element $l_{y'}$ of support of f such that $l_y \leq l_{y'}$.*

Proof. Consider $y' = \frac{y}{p(y)}$. Since $p(y') = 1$ it follows from Proposition (3.2) that $l_{y'}$ is a maximal element and $l_y \leq l'_y$. □

Theorem 3.1. *Let $p, q : X \to \bar{\mathbb{R}}$ be strictly IPH functions. Then the following assertions are equivalent*
(i) $supp(p, L) \subset supp(q, L)$
(ii) For each maximal element l_1 of $supp(p, L)$ there exists a maximal element l_2 of $supp(q, L)$ such that $l_1(x) \leq l_2(x) \quad \forall x \in X$.

Proof. $(i) \Rightarrow (ii)$. Let $supp(p, L) \subset supp(q, L)$. Let l_1 be a maximal element of $supp(p, L)$, so $l_1 \in supp(q, L)$. Then by Proposition (3.3) there exists a maximal element l_2 of $supp(q, L)$ such that $l_1 \leq l_2$.

$(ii) \Rightarrow (i)$ Let, $l \in supp(p, L)$ be arbitary. Then by (3.3) there exists a maximal element l_1 of $supp(p, L)$ such that $l \leq l_1$. Let $l_2 \in supp(q, L)$ and $l \leq l_2$. Then, $l_2 \geq l$, and hence $l \in supp(q, L)$. This completes the proof. □

In the following, we present necessary and sufficient conditions for the minimum of the difference of strictly IPH functions.

Theorem 3.2. *Let $p, q : X \to \bar{\mathbb{R}}$ be strictly IPH functions such that $p(x) \leq q(x)$ for all $x \in X$. Then x_0 is a global minimizer of the function $h = p - q$ if and only if for each $y \in X$ with $0 \neq \tilde{p}(y) = 1$ there exists $z \in X$ whit $q(z) = 1$ such that $l_y \leq l_z$, where $\tilde{p}(x) = p(x) + h(x_0)$ for all $x \in X$.*

Proof. We have that x_0 is a global minimizer of the function h if and only if $supp(\tilde{p}, L) \subset supp(q, L)$. Now the result follows from Theorem (3.1). □

4. Convergence of IPH functions

We need the following well-known definition, A sequence U_k of subsets of X converges to a non-empty set U is the sense of PainleveKuratowski, if U contains all cluster points of all sequences (u_k) with $u_k \in U_k$ and for each $u \in U$ there exists a sequence $u_k \to u$ with $u_k \in U_k (k = 1, ...,)$. The sequence U_k converges to the empty set if each sequence $u_k \in U_k$ has no cluster points.

We shall also use epigraphical convergence (briefly,e-convergence or epi-convergence) of functions. Recall that a sequence of functions $f_k : X \to \bar{\mathbb{R}}$, e-converges to a function f if epi f_k PainleveKuratowski converges to epi f. This means that $\liminf_k f_k(x_k) \geq f(x)$ for each sequence $x_k \to x$ and for every $x \in X$ there exists $x_k \to x$ such that $\limsup_k f_k(x_k) \leq f(x)$. Due to the liminf inequality the latter is equivalent to $lim f_k(x_k) = f(x)$. We shall also use the pointwise convergence: $f_k \to f$ pointwise if $f_k(x) \to f(x)$ for all $x \in X$.

Definition 4.1. Consider a sequence (f_k) of functions defined on X. We say that f_k Li-converges to f if for each $x \in X$ there exists $x_k \to x$ such that $f_k(x_k) \to f(x)$.

Proposition 4.1. *Let $p_k \in \mathcal{P}(X')$. Then p_k Li-converges to $p \in \mathcal{P}(X')$ if and only if p_k pointwise converges to p.*

Proof. We need only to prove that Li-convergence implies pointwise convergence. Let p_k Li-converges to p. Then for each $x \in X'$ there exists a sequence $x_k \to x$ such that $p_k(x_k) \to p(x)$. Since $x_k \to x$ and $x_k \in X'$ it follows that for each $\varepsilon > 0$ and for large enough k it holds that $(1-\varepsilon)x \leq x_k \leq (1+\varepsilon)x$. So by monotonicity of p_k

$$p_k((1-\varepsilon)x) \leq p_k(x_k) \leq p_k((1+\varepsilon)x).$$

Applying positive homogeneity and monotonicity of p_k we get $(1\varepsilon)p_k(x) \leq p_k(x_k) \leq (1+\varepsilon)p_k(x)$, hence

$$\frac{p_k(x_k)}{(1+\varepsilon)} \leq p_k(x) \leq \frac{p_k(x_k)}{(1-\varepsilon)}.$$

Since ε is an arbitrary positive number, we conclude that $p_k(x) \to p(x)$. □

Proposition 4.2. *Let p_k be a sequence of IPH functions defined on X'. Then p_k Li-converges to p if and only if p_k^0 Li-converges to p^0.*

Proof. Recall that by Theorem(2.1) $y_k \to y \iff l_{y_k} \to l_y$. The result follows from the equality $p^0(l_y) = \frac{1}{p(y)}$ in Theorem(2.2). □

Definition 4.2. Let p_k be a sequence of proper IPH functions defined on X and $p \in \mathcal{P}(X)$. We say that p_k $\mathcal{L}$-converges to $p \neq 0$ if p_k Li-converges to p and for each $l \in supp(p, L)$ there exists $l_k \in supp(p_k, L)$ such that $l_k \to l$; p_k $\mathcal{L}$-converges to $p = 0$ if p_k Li-converges to p and each sequence $l_k \in supp(p_k, L)$ has no limit points.

Proposition 4.3. *Let $p_k \in \mathcal{P}(X')$. Then p_k Li-converges to p if and only if p_k $\mathcal{L}$-converges to p.*

Proof. We need to prove only that Li-convergence implies $\mathcal{L}$-convergence. By Proposition (4.1) we can prove that pointwise convergence implies $\mathcal{L}$-convergence. Let p_k pointwise converges to p. Assume that $supp(p, L') \neq \emptyset$ that is $p \neq 0$. We need to prove only that for each $l \in supp(p, L')$ there exists$l^k \longrightarrow l$ with $l^k \in supp(p_k, L')$. Let $l = l_y \in supp(p, L')$. Then by Proposition (2.1) $p(y) \geq 1$. Now assume that $p(y) > 1$. Choose $l_k = l$ for all k. Then $l_k \to l$ and $l_k \in supp(p_k, L')$ for large enough k. Now assume that $p(y) = p^0(l_y) = 1$. Let $y_k = y/p_k(y)$. Since $p_k(y) \to p(y)$ it follows that $y_k \to y$, hence $l_{y_k} \to l_y = l$. We also have $p_k(y_k) = 1$, which implies $l_{y_k} \in supp(p_k, L')$.

Let now $p = 0$. We have to show that each sequence (l_k) with $l_k \in supp(p_k, L')$ has no limit points. Suppose that there is a sequence k_i and a sequence l_{k_i} with $l_{k_i} \in supp(p_{k_i}, L')$ such that $l_{k_i} \to l$. Then $l(x) = \lim l_{k_i}(x) \leq \lim p_{k_i}(x) = 0$ for all $x \in X'$, which is impossible

□

We say that a sequence of proper IPH functions definded on X' converges to p (notation: $p_k \to p$) if p_k converges to p either pointwise or L_i, or epi or $\mathcal{L}$.

Proposition 4.4. *The following assertions are equivalent*

(i) $p_k \to p$;

(ii) $supp(p_k, L') \to supp(p, L')$

Proof. $(i) \Longrightarrow (ii)$.: $Since p_k$ L-converges to p it is enough to show that $l_k \in supp(p_k, L')$, $l_k \to l$ implies $l \in supp(p, L')$. This implication follows directly from the definitions of the support set.

$(ii) \Longrightarrow (i)$. $Let x \in X'$, $\lambda = \limsup_k p_k(x)$. We consider separately the cases $0 < \lambda < +\infty, \lambda = 0$ and $\lambda = +\infty$.

1. Let $0 < \lambda < +\infty$. Assume without loss of generality that $p_k(x) \to \lambda$ and $\lambda = 1$. Let $p_k(x) = \mu_k$ and $\bar{p}_k = (1/\mu_k)p_k$. Then $\bar{p}_k(x) = 1$. Applying Theorem (2.1) we conclude that $l_x \in supp(\bar{p}_k, L')$. Now by positive homogeneity of p_k we have $supp(\bar{p}_k, L') = (1/\mu_k)supp(p_k, L')$, then $supp(\bar{p}_k, L') \to supp(p, L')$, hence $l_x \in supp(p, L')$. Then by Proposition (2.1), $p(x) \geq 1 = \limsup_k p_k(x)$.

Now consider the vector $\bar{x} = x/p(x)$, hence $l_{\bar{x}} \in supp(p, L')$. Since $supp(p_k, L') \to supp(p, L')$, it follows that there exists a sequence $l_k \to l_{\bar{x}}$ such that $l_k \in supp(p_k, L')$. We have $l_k(x) \leq p_k(x)$, hence $p(x) = l_{\bar{x}}(x) = \lim_k l_k(x) \leq \liminf_k p_k(x)$.

2. $\lambda = 0$. This means that $p_k(x) \to 0$, so $p_k(y) \to 0$ for all $y \in X'$. Thus, we need to show that $p = 0$, in other words $supp(p, L') = \emptyset$. Suppose that $supp(p, L')$ is not empty and $l \in supp(p, L')$. Then there exists a sequence $l_k \in supp(p_k, L')$ such that $l_k(y) \to l(y)$ for all y. Since $l_k(y) \leq p_k(y)$ for all y it follows that $l_k \to 0$, so $l = 0$, which is impossible.

3. $\lambda = +\infty$. Without loss of generality assume that $p_k(x) \to +\infty$. Let $y_k := x/p_k(x)$. Then $y_k \to 0$, $l_{y_k} \in supp(p_k, L')$. Let $z \in X'$. There exists k' such that $y_k \leq z$ for all $k \geq k'$. Then $l_z \leq l_{y_k}$, hence $l_z \in supp(p_k, L')$ for $k \geq k'$. Since $supp(p_k, L') \to supp(p, L')$ it follows that $l_z \in supp(p, L')$. We have proved that $supp(p, L') = L'$. □

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