Jordan Journal of Mathematics and Statistics (JJMS) 7(2), 2014, pp.119 - 130

RELATED MONOTONIC FUNCTIONS

M. SASIREKHA

ABSTRACT. We continue the study of related sets introduced by Prof. Min and generalize some of his results.

1. INTRODUCTION

The closure and interior operators in topological spaces are monotonic functions. Prof. *Császár*, in 1997 discussed and studied the properties of monotonic functions defined on the power set of a nonempty set X into itself in his paper [1]. The family of all monotonic functions defined on the power set of a nonempty set X into itself is denoted by $\Gamma(X)$ or simply Γ. The following sub collections of Γ are also defined by *Császár* [1]. $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(\emptyset) = \emptyset\}, \Gamma_1 = \{\gamma \in \Gamma \mid \gamma(X) = X\}, \Gamma_+ = \{\gamma \in \Gamma \mid A \subset \gamma(A)$ for every subset A of $X\}$, $\Gamma_- = \{\gamma \in \Gamma \mid \gamma(A) \subset A$ for every subset A of $X\}$, $\Gamma_{-2} = \{\gamma \in \Gamma \mid \gamma(A) \supset \gamma(\gamma(A))$ for every subset A of $X\}$ and $\Gamma_2 = \{\gamma \in \Gamma \mid \gamma(A) = \gamma(\gamma(A))$ for every subset A of $X\}$. If $\gamma \in \Gamma(X)$, a subset A of X is said to be $\gamma-$open if $A \subset \gamma(A)$. The family of all $\gamma-$open sets, denoted by μ_γ, is a *generalized topology* [2], i.e., $\emptyset \in \mu_\gamma$ and μ_γ is closed under arbitrary union. The pair (X, μ_γ) is called *generalized topological space* or simply a GTS. If A is a subset of X, the largest $\gamma-$open set contained in A, called the $\gamma-$interior of A, is denoted by $i_\gamma(A)$ and the smallest $\gamma-$closed set containing A, called the $\gamma-$closure of A, is denoted by $c_\gamma(A)$. The family of all $\alpha-$open sets, semiopen sets, preopen

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2000 *Mathematics Subject Classification.* 54A05, 54A10.

Key words and phrases. monotonic function, generalized topology, (ι, κ)-related sets.

Received: April 19, 2013 Accepted : March 20 , 2014 .

sets and $\beta-$open sets of the generalized space (X, μ_γ) are also defined and studied by *Császár* [2]. A subset A of X is said to be $\alpha-$open (resp., semiopen, preopen, $\beta-$open) if $A \subset i_\gamma c_\gamma i_\gamma(A)$ (resp., $A \subset c_\gamma i_\gamma(A), A \subset i_\gamma c_\gamma(A), A \subset c_\gamma i_\gamma c_\gamma(A)$).

Lemma 1.1. [3, Theorem 2.4]*Let X be a nonempty set. For $\iota,\ \kappa \in \Gamma$, if ι and κ are $\sigma(\iota,\kappa)-$related and $\pi(\iota,\kappa)-$related, then ι and κ are $\beta(\iota,\kappa)-$related.*

Remark 1. By Lemma 1.1, it follows that if $\iota \in \Gamma$ and $\kappa \in \Gamma$ are not $\beta(\iota,\kappa)-$related, then either ι and κ are not $\sigma(\iota,\kappa)-$related or not $\pi(\iota,\kappa)-$related.

Lemma 1.2. *Let X be a nonempty set and $\kappa \in \Gamma_+$. Then the following hold.*
(a) If $\iota \in \Gamma$, then ι and κ are $\sigma(\iota,\kappa)-$related (resp., $\pi(\iota,\kappa)-$related, $\beta(\iota,\kappa)-$related)[3, Theorem 2.8 (1),(2)and (3)].
(b) If $\iota \in \Gamma$, then ι and κ are $\sigma(\iota,\kappa)-$related, then ι and κ are $\beta(\iota,\kappa)-$related [3, Theorem 2.8(4)].
(c) If $\iota \in \Gamma$, then ι and κ are $\pi(\iota,\kappa)-$related, then ι and κ are $\beta(\iota,\kappa)-$related [3, Theorem 2.8(5)].

Lemma 1.3. [3, Theorem 2.13] *Let X be a nonempty set, $\iota \in \Gamma_2$ and $\kappa \in \Gamma$. If ι and κ are $\sigma(\iota,\kappa)-$related, then they are $\alpha(\iota,\kappa)-$related.*

Lemma 1.4. [3, Theorem 2.14]*Let X be a nonempty set, $\iota \in \Gamma_2$ and $\kappa \in \Gamma$. If ι and κ are $\pi(\iota,\kappa)-$related, then they are $\alpha(\iota,\kappa)-$related.*

2. Related monotonic functions

If X is a nonempty set and $A \subset X$, two functions ι and κ are functions from $\wp(X)$ to $\wp(X)$, then ι and κ are said to be *$\alpha(\iota,\kappa)-$related* (resp., *$\sigma(\iota,\kappa)-$related, $\pi(\iota,\kappa)-$related, $\beta(\iota,\kappa)-$related) for A*, if $\iota(A) \subset \iota\kappa\iota(A)$ (resp., $\iota(A) \subset \kappa\iota(A)$, $\iota(A) \subset \iota\kappa(A)$, $\iota(A) \subset \kappa\iota\kappa(A)$) [3]. $A \subset X$ is said to be $\alpha(\iota,\kappa)-$open (resp., $\sigma(\iota,\kappa)-$open, $\pi(\iota,\kappa)-$open, $\beta(\iota,\kappa)-$open), if $A \subset \iota\kappa\iota(A)$ (resp., $A \subset \kappa\iota(A)$, $A \subset \iota\kappa(A)$, $A \subset \kappa\iota\kappa(A)$). We say that $\iota \in \Gamma_*$ if $\iota(A) \subset \iota^2(A)$ for every subset A of X. Clearly, $\iota \in \Gamma_2$ implies $\iota \in \Gamma_*$ and $\iota \in \Gamma_+$ implies $\iota \in \Gamma_*$. The following Theorem 2.1 discuss the relation between the *related sets and different kind of generalized open sets*. Theorem 2.1(c) is a generalization of Lemma 2.16 of [3].

Theorem 2.1. *Let X be a nonempty set and $\kappa \in \Gamma_+$. Then the following hold.*
(a) If $\iota \in \Gamma_+$ (i.e., every subset is $\iota-$open), then every subset A of X is $\sigma(\iota,\kappa)-$open (resp., $\pi(\iota,\kappa)-$open, $\beta(\iota,\kappa)-$open) and every subset is also $\alpha(\iota,\kappa)-$open.
(b) If $\iota \in \Gamma_\star$, then ι and κ are $\alpha(\iota,\kappa)-$related.
(c) If $\iota \in \Gamma_2$, then $\iota(A)$ is $\alpha(\iota,\kappa)-$open for every subset $A$of X.

Proof. (a) The proof follows from Lemma 1.2(a).
(b)$\kappa \in \Gamma_+$ implies that $\iota(A) \subset \kappa\iota(A)$ for every subset A of X, which implies that $\iota\iota(A) \subset \iota\kappa\iota(A)$ and so $\iota(A) \subset \iota\kappa\iota(A)$, since $\iota \in \Gamma_\star$. Therefore, ι and κ are $\alpha(\iota,\kappa)-$related. Moreover, since $\iota(A) \subset \iota\kappa\iota(\iota(A)), \iota(A)$ is $\alpha(\iota,\kappa)-$open for every subset A of X.
(c) The proof follows from (b). □

Corollary 2.1. *Let (X,μ) be a space and $\iota = i_\mu, \kappa = c_\mu$. Then the following hold.*
(a) Every $\sigma(\iota,\kappa)-$open (resp., $\alpha(\iota,\kappa)-$open, $\pi(\iota,\kappa)-$open, $\beta(\iota,\kappa)-$open) set is a $\sigma(\iota,\kappa)-$related (resp., $\alpha(\iota,\kappa)-$related, $\pi(\iota,\kappa)-$related, $\beta(\iota,\kappa)-$related) set.

The following Example 2.1 shows that the the converse of Lemma 1.2(a) is not true and also, it shows that the condition $\iota \in \Gamma_2$ cannot be dropped in Lemma 1.3. Theorem 2.2 below is a generalization of Lemma 1.1 and the proof is clear.

Example 2.1. *Let $X = \{1,2\}$ and $\iota,\ \kappa : \wp(X) \to \wp(X)$ be defined by $\iota(\emptyset) = \emptyset,\ \iota(\{1\}) = \{1,2\},\ \iota(\{2\}) = \{1\},\ \iota(\{1,2\}) = \{2\}$, κ is the identity function on $\wp(X)$. Then ι and κ are $\sigma(\iota,\kappa)-$related, $\pi(\iota,\kappa)-$related and $\beta(\iota,\kappa)-$related but $\iota \notin \Gamma$. Note that $\iota \notin \Gamma_2$, but we have $\{1\}$ is not $\alpha(\iota,\kappa)-$related, $\{2\}$ is not both $\pi(\iota,\kappa)-$open and $\beta(\iota,\kappa)-$open. Moreover, ι and κ are $\sigma(\iota,\kappa)-$related, $\pi(\iota,\kappa)-$related and $\beta(\iota,\kappa)-$related does not imply that ι and κ are $\alpha(\iota,\kappa)-$related.*

Theorem 2.2. *Let X be a nonempty set, $\kappa \in \Gamma$ and ι is any function from $\wp(X)$ to $\wp(X)$. If ι and κ are $\sigma(\iota,\kappa)-$related and $\pi(\iota,\kappa)-$related, then ι and κ are $\beta(\iota,\kappa)-$related.*

The following Example 2.2(i) and (ii) show that the the converse of Theorem 2.1(a) are not true. Example 2.3 shows that the converse of Theorem 2.1(b) is not true.

Example 2.2. *(i) Let* $X = \{1,2\}$ *and* $\iota,\ \kappa : \wp(X) \to \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset,\ \iota(\{1\}) = \{1,2\},\ \iota(\{2\}) = \{1\},\ \iota(\{1,2\}) = \{1,2\},\ \kappa(\emptyset) = \emptyset,\ \kappa(\{1\}) = \{1\},\ \kappa(\{2\}) = \{1,2\},\ \kappa(\{1,2\}) = \{1,2\}$. *Then every subset is* $\alpha(\iota,\kappa)-$*open,* $\pi(\iota,\kappa)-$*open and* $\beta(\iota,\kappa)-$*open but* $\iota \notin \Gamma_+$. *Note that* $\{2\}$ *is not* $\sigma(\iota,\kappa)-$*open but is* $\sigma(\iota,\kappa)-$*related.*
(ii) Let $X = \{1,2\}$ *and* $\iota,\ \kappa : \wp(X) \to \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset,\ \iota(\{1\}) = \{1,2\},\ \iota(\{2\}) = \{2\},\ \iota(\{1,2\}) = \{1\},\ \kappa(\emptyset) = \emptyset,\ \kappa(\{1\}) = \{1,2\},\ \kappa(\{2\}) = \{2\},\ \kappa(\{1,2\}) = \{1,2\}$. *Then every subset is* $\sigma(\iota,\kappa)-$*open and* $\iota \notin \Gamma_+$.

Example 2.3. *Let* $X = \{1,2\}$ *and* $\iota,\ \kappa : \wp(X) \to \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset,\ \iota(\{1\}) = \{1,2\},\ \iota(\{2\}) = \{1\},\ \iota(\{1,2\}) = \{1,2\},\ \kappa(\emptyset) = \emptyset,\ \kappa(\{1\}) = \{1\},\ \kappa(\{2\}) = \{2\},\ \kappa(\{1,2\}) = \{1,2\}$. ι *and* κ *are both* $\alpha(\iota,\kappa)-$*related and* $\alpha(\iota,\kappa)-$*open but* $\iota \notin \Gamma_\star$.

For ι and $\kappa \in \Gamma,\ \iota$ and κ are said to satisfy condition (R_1)[3] if $\kappa\iota(A) \subset \kappa(A)$ for every subset A of X. ι and κ are said to satisfy condition (R_2) [3] if $\iota\kappa(A) \subset \kappa(A)$ for every subset A of X. The proof of the following Theorem 2.3 is clear. Example 2.4 below shows that the converses of Theorem 2.3 are not true. One can also prove that the converses are true, if κ is a bijection. Example 2.5 below shows that the conditions (R_1) and (R_2) are independent.

Theorem 2.3. *If* ι *and* $\kappa \in \Gamma$ *such that* $\iota \in \Gamma_-$, *then the following hold.*
(a) ι *and* κ *satisfy both* (R_1) *and* (R_2) [3, Lemma 2.10].
(b)ι *and* κ *are* $\alpha(\iota,\kappa)-$*related and so* ι *and* κ *are* $\sigma(\iota,\kappa)-$*related and* $\pi(\iota,\kappa)-$*related* [3, Theorem 2.11].

Example 2.4. *(i) Let* $X = \{1,2\}$ *and* $\iota,\ \kappa : \wp(X) \to \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset,\ \iota(\{1\}) = \{2\},\ \iota(\{2\}) = \{1\},\ \iota(\{1,2\}) = \{1,2\},\ \kappa(\emptyset) = \emptyset,\ \kappa(\{1\}) = \{1\},\ \kappa(\{2\}) = \{1\},\ \kappa(\{1,2\}) = \{1,2\}$.*Then* $\kappa(\{\iota(\{1\})\}) \subset \kappa(\{1\})$ *and* $\kappa(\{\iota(\{2\})\})$

$\subset \kappa(\{2\})$, $\kappa(\{\iota(\{1,2\})\}) \subset \kappa(\{1,2\})$. *Hence* ι *and* κ *satisfies* (R_1) *but* $\iota \notin \Gamma_-$.
(ii) Let $X = \{1,2\}$ *and* $\iota,\ \kappa : \wp(X) \rightarrow \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset$, $\iota(\{1\}) = \{1,2\}$, $\iota(\{2\}) = \{2\}$, $\iota(\{1,2\}) = \{1,2\}$, $\kappa(\emptyset) = \emptyset$, $\kappa(\{1\}) = \{1,2\}$, $\kappa(\{2\}) = \{2\}$, $\kappa(\{1,2\}) = \{1,2\}$. *Then* $\iota(\{\kappa(\{1\})\}) \subset \kappa(\{1\})$ *and* $\iota(\{\kappa(\{2\})\}) \subset \kappa(\{2\})$, $\iota(\{\kappa(\{1,2\})\}) \subset \kappa(\{1,2\})$. *Hence* ι *and* κ *satisfies* (R_2) *but* $\iota \notin \Gamma_-$.

Example 2.5. *(i) Consider* ι *and* κ *of Example 2.4(i). Then* $\iota\kappa(\{2\}) = \{2\} \not\subset \kappa(\{2\})$ *and so* ι *and* κ *does not satisfy* (R_2) *but* ι *and* κ *satisfy* (R_1).
(ii) Let $X = \{1,2\}$ *and* $\iota, \kappa : \wp(X) \rightarrow \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset$, $\iota(\{1\}) = \{1\}$, $\iota(\{2\}) = \{1,2\}$, $\iota(\{1,2\}) = \{1,2\}, \kappa(\emptyset) = \emptyset, \kappa(\{1\}) = \{1,2\}, \kappa(\{2\}) = \{1\}$, $\kappa(\{1,2\}) = \{1,2\}$. *Then* $\{1,2\} = \kappa\iota(\{2\}) \not\subset \kappa(\{2\}) = \{2\}$ *and so* ι *and* κ *does not satisfy* (R_1) *but* ι *and* κ *satisfy* (R_2).

The following Example 2.6 shows that the condition (R_1) in Theorem 2.5(1) of [3] cannot be dropped and the condition (R_2) in Theorem 2.5(2) of [3] cannot be dropped.

Example 2.6. *(i) Consider the functions* ι *and* κ *of Example 2.5(ii). Then* ι *and* κ *does not satisfy* (R_1), *since for* $A = \{2\}, \{1,2\} = \kappa\iota(A) \not\subset \kappa(A) = \{2\}$. *Since* $\{1\} = \iota\kappa(\{2\}) \not\supset \iota(\{2\}) = \{1,2\}$, ι *and* κ *are not* $\pi(\iota,\kappa)-$*related but are* $\alpha(\iota,\kappa)-$*related.*
(ii) Let $X = \{1,2\}$ *and* $\iota, \kappa : \wp(X) \rightarrow \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset$, $\iota(\{1\}) = \{1\}$, $\iota(\{2\}) = \{1\}$, $\iota(\{1,2\}) = \{1,2\}, \kappa(\emptyset) = \emptyset, \kappa(\{1\}) = \{2\}, \kappa(\{2\}) = \{1\}, \kappa(\{1,2\}) = \{1,2\}$. *If* $A = \{1\}$, *then* $\iota\kappa(A) = A$ *but* $\kappa(A) = \{2\}$ *and so* ι *and* κ *does not satisfy* (R_2). *Since* $\kappa\iota(A) = \{2\} \not\supset \iota(A) = A$, ι *and* κ *are not* $\sigma(\iota,\kappa)-$*related but are* $\alpha(\iota,\kappa)-$*related.*

Though the conditions (R_1) and (R_2) are independent for $\iota,\ \kappa \in \Gamma$, the following Theorem 2.4 shows that the conditions are equivalent if $\kappa \in \Gamma_{-2}$, and ι and κ are $\beta(\iota,\ \kappa)-$related.

Theorem 2.4. *If* $\iota,\ \kappa \in \Gamma$ *such that* $\kappa \in \Gamma_{-2}$, *and* ι *and* κ *are* $\beta(\iota,\ \kappa)-$*related, then the conditions* (R_1) *and* (R_2) *are equivalent.*

Proof. Suppose that $\iota,\ \kappa \in \Gamma$ such that $\kappa \in \Gamma_{-2}$, and ι and κ are $\beta(\iota,\ \kappa)-$related. If (R_1) holds, for $A \subset X$, then $\iota\kappa(A) \subset \kappa\iota\kappa(\kappa(A)) \subset \kappa\iota\kappa(A) \subset \kappa\kappa(A) \subset \kappa(A)$ and so $\iota\kappa(A) \subset \kappa(A)$. Therefore, (R_2) holds. Conversely, if (R_2) holds, for $A \subset X$, then $\kappa\iota(A) \subset \kappa(\kappa\iota\kappa(A)) \subset \kappa\iota\kappa(A) \subset \kappa\kappa(A) \subset \kappa(A)$ and so $\kappa\iota(A) \subset \kappa(A)$. Therefore, (R_1) holds. □

The following Example 2.7 shows that the conditions $\kappa \in \Gamma_{-2}$ and ι and κ are $\beta(\iota,\ \kappa)-$related cannot be dropped in the above Theorem 2.4.

Example 2.7. *(i) Consider the monotonic functions ι and κ of Example 2.4(i). Then $\kappa \in \Gamma_{-2}$ and ι and κ satisfy condition (R_1) but not (R_2). But ι and κ are not $\beta(\iota,\ \kappa)-$related, since if $A = \{1\}$, then $\{2\} = \iota(A) \not\subset \kappa\iota\kappa(A) = \{1\}$.*
(ii) Consider the monotonic functions ι and κ of Example 2.5(ii). Then ι and κ are $\beta(\iota,\ \kappa)-$related and ι and κ satisfy condition (R_2) but not (R_1). But $\kappa \notin \Gamma_{-2}$, since if $A = \{2\}$, then $\{1\} = \kappa(A) \not\supset \kappa^2(A) = \{1, 2\}$.

The following Theorem 2.5 is a generalization of Lemma 1.3.

Theorem 2.5. *Let X be a nonempty set, $\iota \in \Gamma_*$ and $\kappa \in \Gamma$. If ι and κ are $\sigma(\iota, \kappa)-$related, then they are $\alpha(\iota, \kappa)-$related.*

Proof. Since for every subset A of X, $\iota(A) \subset \iota^2(A) \subset \iota\kappa\iota(A)$, the proof follows. □

Corollary 2.2. *Let X be a nonempty set, $\iota \in \Gamma_2$ and $\kappa \in \Gamma$. If ι and κ are $\sigma(\iota, \kappa)-$related, then they are $\alpha(\iota, \kappa)-$related* [3, Theorem 2.13].

Corollary 2.3. *Let X be a nonempty set, $\iota \in \Gamma_+$ and $\kappa \in \Gamma$. If ι and κ are $\sigma(\iota, \kappa)-$related, then they are $\alpha(\iota, \kappa)-$related.*

The following Example 2.8 shows that the condition $\iota \in \Gamma_*$ in Theorem 2.5 cannot be dropped.

Example 2.8. *Let $X = \{1, 2\}$ and $\iota, \kappa : \wp(X) \rightarrow \wp(X)$ be defined by $\iota(\emptyset) = \emptyset,\ \iota(\{1\}) = \{2\},\ \iota(\{2\}) = \{1\},\ \iota(\{1, 2\}) = \{1, 2\}, \kappa(\emptyset) = \emptyset, \kappa(\{1\}) = \{1\}, \kappa(\{2\}) =$*

$\{2\}, \kappa(\{1,2\}) = \{1,2\}$. *Then* ι *and* κ *are* $\sigma(\iota,\kappa)-$*related but not* $\alpha(\iota,\kappa)-$*related, since if* $A = \{1\}$, *then* $\{2\} = \iota(A) \not\subset \iota\kappa\iota(A) = \{1\}$. *If* $A = \{1\}$, *then* $\{2\} = \iota(A) \not\subset \iota^2(A) = \{1\}$ *and so* $\iota \notin \Gamma_\star$.

The following Theorem 2.6 is a generalization of Lemma 1.4.

Theorem 2.6. *Let* X *be a nonempty set,* $\iota \in \Gamma_*$ *and* $\kappa \in \Gamma$. *If* ι *and* κ *are* $\pi(\iota,\kappa)-$*related, then they are* $\alpha(\iota,\kappa)-$*related.*

Proof. Since for every subset A of X, $\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota\kappa(\iota(A)) = \iota\kappa\iota(A)$, the proof follows. □

Corollary 2.4. *Let* X *be a nonempty set,* $\iota \in \Gamma_2$ *and* $\kappa \in \Gamma$. *If* ι *and* κ *are* $\pi(\iota,\kappa)-$*related, then they are* $\alpha(\iota,\kappa)-$*related* [3, Theorem 2.14].

Corollary 2.5. *Let* X *be a nonempty set,* $\iota \in \Gamma_+$ *and* $\kappa \in \Gamma$. *If* ι *and* κ *are* $\pi(\iota,\kappa)-$*related, then they are* $\alpha(\iota,\kappa)-$*related.*

The following Example 2.9 shows that the condition $\iota \in \Gamma_*$ in Theorem 2.6 cannot be dropped.

Example 2.9. *Consider the monotonic functions* ι *and* κ *in Example 2.8. Then* ι *and* κ *are* $\pi(\iota,\kappa)-$*related but not* $\alpha(\iota,\kappa)-$*related, since if* $A = \{2\}$, *then* $\{1\} = \iota(A) \not\subset \iota\kappa\iota(A) = \{2\}$. *Since if* $A = \{1\}$, *then* $\{2\} = \iota(A) \not\subset \iota^2(A) = \{1\}$ *and so* $\iota \notin \Gamma_\star$.

The following Theorem 2.7 is a generalization of Theorem 2.15 of [3].

Theorem 2.7. *Let* X *be a nonempty set,* $\iota \in \Gamma_*$ *and* $\kappa \in \Gamma$. *Then the following hold*
(a) If ι *and* κ *are* $\sigma(\iota,\kappa)-$*related and* (R_1), *then they are* $\pi(\iota,\kappa)-$*related.*
(b) If ι *and* κ *are* $\pi(\iota,\kappa)-$*related and* (R_2), *then they are* $\sigma(\iota,\kappa)-$*related.*

Proof. (a) Since for every subset A of X, $\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota(\kappa\iota(A)) \subset \iota\kappa(A)$, the proof follows.
(b) Since for every subset A of X, $\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota\kappa(\iota(A)) \subset \kappa\iota(A)$, the proof follows. □

Corollary 2.6. *Let X be a nonempty set, $\iota \in \Gamma_2$ and $\kappa \in \Gamma$. Then the following hold* [3, Theorem 2.15].
(a) If ι and κ are $\sigma(\iota,\kappa)-$related and (R_1), then they are $\pi(\iota,\kappa)-$related.
(b) If ι and κ are $\pi(\iota,\kappa)-$related and (R_2), then they are $\sigma(\iota,\kappa)-$related.

Corollary 2.7. *Let X be a nonempty set, $\iota \in \Gamma_+$ and $\kappa \in \Gamma$. Then the following hold.*
(a) If ι and κ are $\sigma(\iota,\kappa)-$related and (R_1), then they are $\pi(\iota,\kappa)-$related.
(b) If ι and κ are $\pi(\iota,\kappa)-$related and (R_2), then they are $\sigma(\iota,\kappa)-$related.

The following Theorem 2.8 is a generalization of Lemma 2.16 of [3].

Theorem 2.8. *Let X be a nonempty set, $\iota \in \Gamma_*$ and $\kappa \in \Gamma_+$. Then ι and κ are $\alpha(\iota,\kappa)-$related.*

Proof. Since for every subset A of X, $\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota(\kappa(\iota(A)))$, the proof follows. □

Corollary 2.8. *Let X be a nonempty set, either $\iota \in \Gamma_2$ or $\iota \in \Gamma_+$ and $\kappa \in \Gamma_+$. Then ι and κ are $\alpha(\iota,\kappa)-$related* [3, Lemma 2.16].

The following Theorem 2.9 is a generalization of Lemma 2.18 of [3].

Theorem 2.9. *Let X be a nonempty set, $\iota \in \Gamma$ and $\kappa \in \Gamma_{-2}$. If $\iota(A) \subset \kappa(A)$, then ι and κ satisfy both (R_1) and* (R_2).

Proof. Since for every subset A of X, $\kappa\iota(A) \subset \kappa\kappa(A) \subset \kappa(A)$, and so $\kappa\iota(A) \subset \kappa(A)$. Therefore (R_1) holds.
Since for every subset A of X, $\iota\kappa(A) \subset \kappa\kappa(A) \subset \kappa(A)$, and so $\iota\kappa(A) \subset \kappa(A)$. Therefore (R_2) holds. □

Corollary 2.9. [3, Lemma 2.18] *Let X be a nonempty set, $\iota \in \Gamma$ and $\kappa \in \Gamma_2$. If $\iota(A) \subset \kappa(A)$, then ι and κ satisfy (R_1) and* (R_2).

The following Example 2.11 shows that the converse of Theorem 2.9 is not true.

Example 2.10. *Let* $X = \{1,2\}$ *and* $\iota, \kappa : \wp(X) \to \wp(X)$ *be defined by* $\iota(\emptyset) = \emptyset$, $\iota(\{1\}) = \{1\}$, $\iota(\{2\}) = \{2\}$, $\iota(\{1,2\}) = \{1,2\}, \kappa(\emptyset) = \emptyset, \kappa(\{1\}) = \{1\}, \kappa(\{2\}) = \{1\}, \kappa(\{1,2\}) = \{1,2\}$. *Then* ι *and* κ *satisfy* (R_1),(R_2)*and* $\kappa \in \Gamma_{-2}$. *But* $\iota(A) \not\subset \kappa(A)$, *since if* $A = \{1\}$, *then* $\{2\} = \iota(A) \not\subset \kappa(A) = \{1\}$.

The following Theorem 2.10 is a generalization of Lemma 2.19 of [3]. Moreover, it also shows that Lemma 2.18(2) of [3] is valid without the condition that $\iota(A) \subset \kappa(A)$ for every subset A of X.

Theorem 2.10. *Let* X *be a nonempty set,* $\iota \in \Gamma$ *and* $\kappa \in \Gamma_{-2}$. *If* ι *and* κ *satisfy either* (R_1)*or* (R_2), *then* $\kappa\iota\kappa(A) \subset \kappa(A)$ *for every* $A \subset X$.

Proof. For every subset A of X, $\kappa\iota\kappa(A) \subset \kappa\kappa(A) \subset \kappa(A)$ and so $\kappa\iota\kappa(A) \subset \kappa(A)$. □

Corollary 2.10. [3, Lemma 2.19] *Let* X *be a nonempty set,* $\iota \in \Gamma$ *and* $\kappa \in \Gamma_2$. *If* ι *and* κ *satisfy either* (R_1)*or* (R_2), *then* $\kappa\iota\kappa(A) \subset \kappa(A)$ *for every* $A \subset X$.

The following Theorem 2.11 is a generalization of Lemma 2.20 of [3] which shows that the condition $\kappa \in \Gamma_2$ is redundant. Moreover, the following Example 2.11 shows that the condition $\sigma(\iota,\kappa)-$related in Theorem 2.11(a) and the condition $\pi(\iota,\kappa)-$related in Theorem 2.11(b) cannot be dropped.

Theorem 2.11. *Let* X *be a nonempty set and* $\iota, \kappa \in \Gamma$. *Then the following hold.*
(a) If ι *and* κ *satisfy* (R_1) *and* $\sigma(\iota,\kappa) - related$, *then* $\iota(A) \subset \kappa(A)$.
(b) If ι *and* κ *satisfy* (R_2) *and* $\pi(\iota,\kappa) - related$, *then* $\iota(A) \subset \kappa(A)$.

Proof. (a) For every subset A of X, $\iota(A) \subset \kappa\iota(A) \subset \kappa(A)$. Therefore $\iota(A) \subset \kappa(A)$.
(b) For every subset A of X, $\iota(A) \subset \iota\kappa(A) \subset \kappa(A)$. Therefore $\iota(A) \subset \kappa(A)$. □

Example 2.11. *(a) The functions* ι *and* κ *in Example 2.4(i) satisfy* (R_1) *but not* $\sigma(\iota,\kappa)-$*related, since* $\{2\} = \iota(\{1\}) \not\subset \kappa\iota(\{1\}) = \{1\}$. *Note that* $\iota(\{1\}) \not\subset \kappa(\{1\})$.
(b) The functions ι *and* κ *in Example 2.5(ii) satisfy* (R_2) *but not* $\pi(\iota,\kappa)-$*related, since* $\{1,2\} = \iota(\{2\}) \not\subset \iota\kappa(\{2\}) = \{1\}$. *Note that* $\iota(\{2\}) \not\subset \kappa(\{2\})$.

The following Theorem 2.12 is a generalization of Lemma 2.21 of [3] and Lemma 1.13 of [1].

Theorem 2.12. *Let X be a nonempty set, $\iota \in \Gamma_*, \kappa \in \Gamma_{-2}$. If ι and κ are $\sigma(\iota, \kappa)$ related and satisfy either (R_1) or (R_2), then the following hold.*
(a)$\iota\kappa\iota\kappa(A) = \iota\kappa(A)$.
(b)$\kappa\iota\kappa\iota(A) = \kappa\iota(A)$.
(c)$(\iota\kappa\iota)(\iota\kappa\iota)(A) = (\iota\kappa\iota)(A)$.
(d)$(\kappa\iota\kappa)(\kappa\iota\kappa)(A) = (\kappa\iota\kappa)(A)$.
(e) If γ is a product of alternating factors ι and κ, then $\gamma \in \Gamma_2$.

Proof. (a) For every subset A of X,$\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota\kappa\iota(A)$ and so $\iota\kappa(A) \subset \iota\kappa\iota\kappa(A)$. Again, $\iota\kappa\iota\kappa(A) \subset \iota\kappa\kappa(A) \subset \iota\kappa(A)$ and so $\iota\kappa\iota\kappa(A) \subset \iota\kappa(A)$. Hence $\iota\kappa\iota\kappa(A) = \iota\kappa(A)$.
(b) For every subset A of X,$\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota\kappa\iota(A)$ which implies that $\kappa\iota(A) \subset \kappa\iota\kappa\iota(A)$. Again, $\kappa\iota\kappa\iota(A) \subset \kappa\kappa\iota(A) \subset \kappa\iota(A)$ and so $\kappa\iota\kappa\iota(A) \subset \kappa\iota(A)$. Hence $\kappa\iota\kappa\iota(A) = \kappa\iota(A)$.
(c) For every subset A of X,$\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota\kappa\iota(A) \subset \iota\iota\kappa\iota(A)$ which implies that $\kappa\iota(A) \subset \kappa\iota\iota\kappa\iota(A)$ and so $\iota\kappa\iota(A) \subset (\iota\kappa\iota)(\iota\kappa\iota)(A)$. Again, $(\iota\kappa\iota)(\iota\kappa\iota)(A) \subset (\iota\kappa(\iota\kappa\iota))(A) = (\iota\kappa\iota\kappa\iota)(A) \subset \iota\kappa\kappa\iota(A) \subset \iota\kappa\iota(A)$ which implies that $(\iota\kappa\iota)(\iota\kappa\iota)(A) \subset (\iota\kappa\iota)(A)$. Hence $(\iota\kappa\iota)(\iota\kappa\iota)(A) = (\iota\kappa\iota)(A)$.
(d) For every subset A of X,$\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \kappa\iota\iota(A) \subset \kappa\iota\kappa\iota(A) \subset \kappa\iota\kappa\kappa\iota(A)$ which implies that $\iota\kappa(A) \subset \kappa\iota\kappa\kappa\iota\kappa(A)$ which in turn implies that $\kappa\iota\kappa(A) \subset \kappa\kappa\iota\kappa\kappa\iota\kappa(A) \subset \kappa\iota\kappa\kappa\iota\kappa(A)$ and so $\kappa\iota\kappa(A) \subset (\kappa\iota\kappa)(\kappa\iota\kappa)(A)$. Again, $(\kappa\iota\kappa)(\kappa\iota\kappa)(A) \subset \kappa\iota\kappa\iota\kappa(A) \subset \kappa\kappa\iota\kappa(A) \subset \kappa\iota\kappa(A)$ which implies that $(\kappa\iota\kappa)(\kappa\iota\kappa)(A) \subset \kappa\iota\kappa(A)$. Hence $(\kappa\iota\kappa)(\kappa\iota\kappa)(A) = \kappa\iota\kappa(A)$.
(e) The proof is clear. □

Corollary 2.11. [3, Lemma 2.21] *Let X be a nonempty set, $\iota, \kappa \in \Gamma_2$.If ι and κ are $\sigma(\iota, \kappa)$ related and satisfy (R_1), then the following hold.*

(a)$\iota\kappa\iota\kappa(A) = \iota\kappa(A)$.

(b)$\kappa\iota\kappa\iota(A) = \kappa\iota(A)$.

The following Example 2.12 shows that the conditions (R_1) or (R_2) cannot be dropped in Theorem 2.12.

Example 2.12. *(i) Consider the functions* ι *and* κ *defined in Example 2.2(i).* $\iota \in \Gamma_\star$, $\kappa \in \Gamma_{-2}$ *and* ι *and* κ *are* $\sigma(\iota,\kappa)-$*related. If* $A = \{1\}, \{1,2\} = \kappa\iota(A) \not\subset \kappa(A) = \{1\}$ *and so* ι *and* κ *does not satisfy* (R_1)*, and also* $\kappa\iota \notin \Gamma_2$.
(ii) Consider the functions ι *and* κ *defined in Example 2.3.* $\iota \in \Gamma_\star, \kappa \in \Gamma_{-2}$*, and* ι *and* κ *are* $\sigma(\iota,\kappa)-$*related. If* $A = \{1\}$*, then* $\{1,2\} = \iota\kappa(A) \not\subset \kappa(A) = \{1\}$ *and so* ι *and* κ *does not satisfy* (R_2)*, and also* $\iota\kappa \notin \Gamma_2$.

The following Theorem 2.13 is a generalization of Lemma 2.22 of [3].

Theorem 2.13. *Let* X *be a nonempty set,* $\iota \in \Gamma_*, \kappa \in \Gamma_{-2}$*.If* ι *and* κ *are* $\beta(\iota,\kappa)$ *related and satisfy either* (R_1)*or* (R_2)*, then the following hold.*
(a) ι *and* κ *are* $\sigma(\iota,\kappa) - related$
(b) ι *and* κ *are* $\pi(\iota,\kappa)$*-related.*
(c) ι *and* κ *are* $\alpha(\iota,\kappa)$*-related.*

Proof. (a) For every subset A of X, $\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \kappa\iota\kappa(\iota(A))) \subset \kappa\kappa\iota(A) \subset \kappa\iota(A)$. Therefore $\iota(A) \subset \kappa\iota(A)$.
(b) For every subset A of X, $\iota(A) \subset \iota^2(A) = \iota(\iota(A)) \subset \iota(\kappa\iota\kappa(A)) = \iota\kappa(A)$, by (a). Therefore $\iota(A) \subset \iota\kappa(A)$.
(c) The proof follows from Theorem2.6 □

Corollary 2.12. [3, Lemma 2.22] *Let* X *be a nonempty set,* $\iota, \kappa \in \Gamma_2$*. If* ι *and* κ *are* $\beta(\iota,\kappa)$ *related and satisfy either* (R_1) *or* (R_2)*, then the following hold.*
(a) ι *and* κ *are* $\sigma(\iota,\kappa) - related$
(b) ι *and* κ *are* $\pi(\iota,\kappa)$*-related.*
(c) ι *and* κ *are* $\beta(\iota,\kappa)$*-related.*

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Department of Mathematics, GKM College of Engineering and Technology, Perungalathur, Chennai-600 063,Tamil Nadu, India.

E-mail address: sasirekha.22@gmail.com