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CENTRAL IDEMPOTENT OF RINGS

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ABSTRACT: In this paper, we find several necessary conditions for the idempotents of a ring R to be central (for example: if $eU_R = U_R e$ for every idempotent e of R then the idempotents of R are central, where U_R is the set of units in R).We present some several properties of a ring whose idempotents are central (for instance: If $I_R \subseteq C_R$ then $reg\ R = S.cl_R$ where $reg\ R$ is the set of regular elements in R and $S.cl_R$ is the set of strongly clean element in R).

1. INTRODUCTION

Idempotent elements in rings are important concepts that contribute deeply and widely in Ring Theory. The central idempotents has drew the attention of many researchers in Ring Theory. For example as in [1], [2], and [6]. In this paper we discuss many necessary conditions that make all idempotents of a ring central. Then we use central idempotents to obtain many properties for rings which possess central idempotents such as linking different rings together. Although the proofs look straight forward, the results merit to be recorded.

2. DEFINITIONS AND BASIC PROPERTIES

We present in this section the notations and the necessary definitions of terms used in this paper.

Also, we give some known or simple results which are useful in the next sections.

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Notation 1.1 Throughout this paper the following notations will be adopted.

R: Associative ring with unity.	U_R : Group of units in R.
If A is a subset of R, then $\overline{A}$ denote the set $R \setminus A$	P_R : Set of all periodic elements in R.
C_R : Center of R.	T_R : Set of all torsion elements of R.
I_R : Set of all idempotent elements of R.	ST_R : Which is $\{x \in R\,; x^2 = 1\}$ the set of all simple torsion elements of R.
N_R : Set of all nilpotent elements of R.	$reg\ R$: Which is $\{x \in R\,; \exists\, y \in R;\ x = x\,y\,x\}$ the set of all regular elements of R.
SN_R : Which is $\{x \in R\,; x^2 = 0\}$ the set of all simple nilpotent elements of R.	$\pi - reg\ R$: which is the set of all π-regular elements of R.
$S_t N_R$: Set of all strongly nilpotent elements of R.	cl_R : Which is $\{x \in R\,; \exists u \in U_R\ \&\ e \in I_R\ ;\ x = u + e\}$ the set of all clean elements of R.
Z_R : Set of all zero divisor of R.	$S.cl_R$: Which is $\{x \in R\,; \exists u \in U_R\ \&\ e \in I_R\ ;\ x = u \cdot e = e \cdot u\}$ the set of all strongly clean elements of R.
$rad\ R$: Prime radical of R.	$A.cl_R$: Which is $\{x \in R; \exists e \in I_R\ \&\ r \in \overline{Z}_R\ ; x = e + r\}$ the set of all almost clean elements of R.
J_R : Jacobson radical of R.	ϕ : The empty set.

Notes 1.2 We can see easily that:

1) $SN_R \cap U_R = \phi$, so $SN_R \subseteq \overline{U_R}$.

2) $I_R \cap SN_R = \{0\}$, so $I_R \subseteq \overline{SN_R} \cup \{0\}$ and $SN_R \subseteq \overline{I_R} \cup \{0\}$.

3) $I_R \cap U_R = \{1\}$, so $I_R \subseteq \overline{U_R} \cup \{1\}$ and $U_R \subseteq \overline{I_R} \cup \{1\}$.

4) If $e \in I_R$, then $(2e-1)^2 = 1$, so $(2e-1) \in ST_R$.

Definitions 1.3

- A ring R is said to be semi commutative if $aR = Ra \quad \forall a \in R$.
- A ring R is said to be regular ring if $reg\ R = R$.
- A right ideal I of a ring R is right pure if for all $a \in I, \exists b \in I$ such that $a = ab$.

The ring R is called a fully right pure ring if each proper right ideal of R is right pure.

- An element a of a ring R is said to be a right (left) strongly regular if there exists an element b of R such that $a = a^2b\left(a = ba^2\right)$. A ring R is said to be a right (left) strongly regular ring if all its elements are right (left) a strongly regular. R is a strongly regular ring if it is both right and left strongly regular.
- A ring R is called a zero-insertive ring if for every $a, b \in R$ with $ab = 0$, then $arb = 0 \quad \forall r \in R$.
- A duo- ring is a ring in which every one sided ideal is a two sided ideal.
- A ring R is said to be a clean ring if $cl_R = R$.
- A ring R is said to be a strongly clean ring if $S \cdot cl_R = R$.
- A ring R is said to be an almost clean ring if $A \cdot cl_R = R$.
- A proper right ideal P of R is said to be a generalized right primary right ideal if

for any two right ideals I, J of R such that $IJ \subseteq P$, we have either $I \subseteq P$ or $\exists n \in N$ such that $J^n \subseteq P$.

- A ring R is said to be a generalized right primary right ring if $\{0\}$ is a generalized right primary right ideal.
- A ring R is said to be a fully generalized right primary right ring if each proper right ideal of R is a generalized right primary right ideal in R.

Lemma 1.4 [7] If one of the following conditions is satisfied, then $I_R \subseteq C_R$:

1) R Is a semi commutative ring.
2) $SN_R = \{0\}$.
3) $e \cdot e' = e' \cdot e \quad \forall e, e' \in I_R$.
4) $eR = Re \quad \forall e \in I_R$.
5) R is a strongly regular ring.
6) R is a zero- insertive ring.
7) R is a duo-ring.
8) R is a local ring.

Proof: We give the proof of 3) because this proof will occur several times in the article.

Let $x = e + ea - eae$ & $y = e + ae - eae$, then $x, y \in I_R$.

Thus,

$ex = xe \Rightarrow e(e+ea-eae) = (e+ea-eae)e \Rightarrow ea = eae \ldots (1)$

$ey = ye \Rightarrow e(e+aea-eae) = (e+ae-eae)e \Rightarrow ae = eae \ldots (2)$

From (1) & (2) we conclude that $ae = ea \quad \forall a \in R$. Therefore $I_R \subseteq C_R$.

Corollary 1.5

a) If R is a semi prime ring and the elements of SN_R are right semi commutative [i.e., $Ra \subseteq aR$ $\forall$ $a \in SN_R$] , then $I_R \subseteq C_R$.

b) If R is a fully right pure ring, then $I_R \subseteq C_R$.

Proof:

a) Let $a \in SN_R$, then $Ra \subseteq aR$ and hence $aRa \subseteq a^2R = \{0\}$. Since $\{0\}$ is a semi prime ideal, we get $a \in \{0\}$. Thus $SN_R = \{0\}$. By (2) of Lemma 1.4 $I_R \subseteq C_R$.

b) Let $a \in SN_R$, then aR is a right ideal of R and $a \in aR$. Since aR is a right pure ideal, there exists an element b of aR such that $a = ab$, but $b \in aR$ implies $b = ar; r \in R$. So $a = a^2r = 0r = 0$.

Thus $SN_R = \{0\}$. By Lemma 1.4 $I_R \subseteq C_R$.

Lemma 1.6 If R is a regular ring, then the following conditions are equivalent:

1) $I_R \subseteq C_R$.
2) $SN_R = \{0\}$.
3) $N_R = \{0\}$.

Proof: The proof is very easy.

Lemma 1.7 Let R be a right strongly regular ring. Then $I_R \subseteq C_R$. Moreover, if $a = a^2b$ for some $b \in R$, then $a = ba^2$, ab = ba, and R is a strongly regular ring.

Proof: It is clear that $SN_R = \{0\}$ and hence $I_R \subseteq C_R$ (by Lemma 1.4).

Let $a \in R$ such that $a = a^2b$ .Then $(a-aba)^2 = 0$. Hence $(a-aba) \in SN_R = \{0\}$, which implies $a = aba$. It follows that ab and ba belong to $I_R \subseteq C_R$. So, $a = aba = ba^2$, which yields $ab = ba^2b = ba$.

Lemma 1.8 Let $a \in R$. The following conditions are equivalent:

1) There exists $b \in R$ such that $a = a^2b$ and $ab = ba$.
2) The element a is a strongly clean element.

Proof: 1 $\Rightarrow$ 2: By hypotheses we have $a = aba$. Let $c = bab$ then we can easily see that $a = aca, c = cac$, and $ac = ca$. So $a = a^2c = ca^2$ and $c = c^2a = ac^2$.

Thus, $(1-ac+a)(1-ac+c) = (1-ac+c)(1-ac+a) = 1$ which means that $u = (1-ac+a)$ is a unit. Denote the idempotent ac by e.

We have, $a = ac - ac + a = ac - (ac)^2 + a^2c = ac(1-ac+a) = (1-ac+a)ac$ or, $a = eu = ue$. This means that a is strongly clean.

2 $\Rightarrow$ 1: Let a be a strongly clean element of R then there exist $u \in U_R$ and $e \in I_R$ such that $a = eu = ue$. So $a^2 = ue \cdot eu = ueu$, and hence $a^2u^{-1} = u^{-1}a^2 = ue = eu = a$. Let $b = u^{-1}$. Then $a = a^2b$ and $ab = au^{-1} = e = u^{-1}a = ba$.

Lemma 1.9 If $\overline{U_R} \subseteq SN_R$, then $I_R \subseteq C_R$.

Proof: It is enough to proof that $\overline{U_R} \subseteq J_R$ to conclude that R is a local ring,(proposition 1, of [8])so $I_R = \{0,1\}$. Let $0 \neq a \in \overline{U_R}$ then, by hypotheses, $a \in SN_R$, so $a^2 = 0$ and $a \neq 0$. Let $s \in R$ and put $x = as$ then $ax = a \cdot as = a^2s = 0$ so x is not right invertible, so $x \notin U_R$, therefore $x \in \overline{U_R} \subseteq SN_R$. Hence $x^2 = 0$, this implies $(1-x)(1+x) = 1$, i.e., $1 - as = 1 - x$ is right invertible for all s of R. This means that $a \in J_R$[proposition 3, of [8]].Thus R is a local ring. So, $I_R = \{0,1\} \subseteq C_R$.

Remark 1.10 In Lemma 1.9, we can use N_R instead of SN_R.

3. NECESSARY CONDITIONS ON RINGS THAT IMPLY $I_R \subseteq C_R$.

We find in this section several necessary conditions in order for the set of all idempotents of a ring R be included in the centre of R.

Lemma 2.1 Let $A \subseteq R$ and $e \in I_R$. If $e \cdot A = A \cdot e$, then $e.a = a.e \quad \forall a \in A$.

Proof: Let $a \in A$ and $e \in I_R$.Then $ea \in e \cdot A = A \cdot e$ so there exists a'∈A such that $e \cdot a = a' \cdot e$.

So $eae = a'ee = a'e = ea \ldots (1)$

Also, $ae \in Ae = eA$ so there exists $a'' \in A$ such that $ae = ea''$ which implies $eae = eea'' = ea'' = ae \ldots (2)$

From (1) & (2) we conclude that $ae = ea$.

Corollary 2.2 a) If $e \cdot I_R = I_R \cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.

b) Let $A = I_R \setminus C_R$. If $e \cdot A = A \cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.

Proof: a) Let $e \cdot I_R = I_R \cdot e \quad \forall e \in I_R$. By Lemma 2.1 , $e \cdot e' = e' \cdot e \quad \forall e, e' \in I_R$.

So, by 3) of Lemma 1.4, $I_R \subseteq C_R$.

b) Let e, e' be elements of I_R. If $e' \in C_R$, then $e \cdot e' = e' \cdot e$, if $e' \notin C_R$, then $e' \in A$.

Since $e \cdot A = A \cdot e$, by Lemma 2.1, $e \cdot e' = e' \cdot e$. Therefore, by 3) of Lemma 1.4, $I_R \subseteq C_R$.

The following Lemma is a generalization of Theorem 1 of [6].

Lemma 2.3 a) If the elements of SN_R commute with the elements of I_R, then $I_R \subseteq C_R$.

b) If the element of SN_R commute with the elements of ST_R in a ring R in which $2x = 0 \Rightarrow x = 0$, then $I_R \subseteq C_R$.

Proof: a) Let $r \in R$ and $e \in I_R$. Then $(re - ere) \in SN_R$. By hypotheses,

$$(re - ere)e = e(re - ere) = 0 \Rightarrow re = ere \ldots (1)$$

Also, $er - ere \in SN_R$. Similarly, $e(re - ere) = (re - ere)e = 0 \Rightarrow er = ere \ldots (2)$

From (1) & (2) we obtain $er = re \quad \forall r \in R$, or equivalently $I_R \subseteq C_R$.

b) Let $e \in I_R$. Then $(2e - 1) \in ST_R$. By hypotheses, for each s of SN_R we have

$$s(2e-1) = (2e-1)s \Rightarrow 2se = 2es \Rightarrow 2(se - es) = 0 \Rightarrow se - es = 0 \Rightarrow es = se.$$

By (a) of this Lemma, we get $I_R \subseteq C_R$.

Corollary 2.4 It results from Lemma 2.3, that**:**

1) If R is a ring in which $ST_R \subseteq C_R$ (in particular, $ST_R = \{1\}$ when R is a torsion free ring) and $2x = 0 \Rightarrow x = 0$, then $I_R \subseteq C_R$.

2) If R is a ring in which $SN_R \subseteq C_R$ (in particular, $SN_R = \{0\}$ when R is a reduced ring) then $I_R \subseteq C_R$.

Corollary 2.5

a) If $e\cdot SN_R = SN_R \cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.
b) If $e.\overline{SN_R} = \overline{SN_R}.e \;\; \forall e\in I_R$, then $I_R \subseteq C_R$.
c) If $e.\overline{I_R} = \overline{I_R}.e \;\; \forall e\in I_R$, then $I_R \subseteq C_R$.
d) Let $A = SN_R \setminus C_R$. If $e\cdot A = A\cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.

Proof: a) By Lemma 2.1, we have $e\cdot s = s\cdot e$, for all $s\in SN_R$ and all $e\in I_R$. So by Lemma 2.3 we have $I_R \subseteq C_R$.

b) Assume $e.\overline{SN_R} = \overline{SN_R}.e$ for every $e\in I_R$. By Lemma 2.1, $e\cdot x = x\cdot e \quad \forall x\in \overline{SN_R}$. Since $I_R \setminus \{0\} \subseteq \overline{SN_R}$, We have $e\cdot e' = e'\cdot e \quad \forall e, e'\in I_R$. By Lemma 1.2 $I_R \subseteq C_R$.

c) Assume $e\cdot \overline{I_R} = \overline{I_R}\cdot e$ for every $e\in I_R$. By Lemma 2.1, $e\cdot x = x\cdot e \quad \forall x\in \overline{I_R}$. Since $SN_R \setminus \{0\} \subseteq \overline{I_R}$, we have $e\cdot s = s\cdot e \quad \forall s\in SN_R$, By part (a) of this Corollary, $I_R \subseteq C_R$.

The proof of d) is similar to the proof of b) of Corollary 2.2.

Remark 2.6 In Corollary 2.5 , we can use N_R in place of SN_R.

Proposition 2.7

a) If $e\cdot U_R = U_R \cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.
b) If $e\cdot \overline{U_R} = \overline{U_R}\cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.
c) If the group U_R is commutative in a ring R in which $2x = 0 \Rightarrow x = 0$, then $I_R \subseteq C_R$.
d) If the elements of ST_R commutes with itself in a ring R in which $4x = 0 \Rightarrow x = 0$, then $I_R \subseteq C_R$.
e) Let $A = U_R \setminus C_R$. If $e\cdot A = A\cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.
f) Let $B = \overline{U_R} \setminus C_R$. If $e\cdot B = B\cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.

Proof: a) $e\cdot U_R = U_R \cdot e$ implies, by Lemma 2.1, $e\cdot u = u\cdot e \quad \forall u\in U_R$.

Let $s\in SN_R$, then $s^2 = 0$, so $(1-s)(1+s) = 1$, i.e., $(1-s)\in U_R$ thus $e(1-s) = (1-s)e$, or $es = se \quad \forall\, s\in SN_R$ and $e\in I_R$, therefore, by Lemma 2.3 , we have $I_R \subseteq C_R$.

b) Apply the fact that $I_R \subseteq \overline{U_R}$ along with part (a) of Corollary 2.2.

c) Let $e \in I_R$ and $s \in SN_R$ then $(s-1)$ and $(2e-1)$ belong to U_R. The equation $(2e-1)\cdot(s-1)=(s-1)\cdot(2e-1)$ implies $2(es-se)=0$, which gives in turn that $e\cdot s = s\cdot e$ therefore, by Lemma 2.3 , we have $I_R \subseteq C_R$.

d) Let $e,e' \in I_R$, then $(2e-1)$ and $(2e'-1)$ belong to ST_R.

Thus, $(2e-1)\cdot(2e'-1)=(2e'-1)\cdot(2e-1) \Rightarrow 4(ee'-e'e)=0 \Rightarrow ee'=e'e \quad \forall e,e' \in I_R$.

By 3) of Lemma 1.4 $I_R \subseteq C_R$.

The proofs of e) and f) are similar to the proof of b) of Corollary 2.2.

Remark 2.8 It follows from the above proposition that if $I_R \not\subset C_R$, then $\exists\, e \in I_R$ and $u \in U_R$ such that $eu \neq ue$. This means that R is not a strongly clean ring. So if R is strongly clean then $I_R \subseteq C_R$.

Proposition 2.9

a) If $e\cdot regR = regR\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.
b) If $e\cdot(\pi - regR) = (\pi - regR)\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.
c) Let $A = regR \setminus C_R$. If $e\cdot A = A\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.
d) Let $B = (\pi - regR)\setminus C_R$. If $e\cdot B = B\cdot e \quad \forall e \in I$, then $I_R \subseteq C_R$.

Proof: a) Assume $e\cdot reg\,R = reg\,R\cdot e$ for every e of I_R. By Lemma 2.1, $e\cdot x = x\cdot e \quad \forall x \in reg\,R$. Since $I_R \subseteq regR$, we have $e\cdot e' = e'\cdot e$ for all e and e' of I_R, By 3) of Lemma 1.4, $I_R \subseteq C_R$.

b) This follows from the fact that $reg\,R \subseteq \pi - reg\,R$ and Lemma 2.1.

The proofs of c) and d) are similar to the proof of b) in Corollary 2.2.

Proposition 2.10

a) If $e\cdot cl_R = cl_R\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.

b) If $e\cdot(S.cl_R) = (S.cl_R)\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.

c) Let $A = cl_R \setminus C_R$. If $e\cdot A = A\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.

d) Let $B = (S.cl_R)\setminus C_R$. If $e\cdot B = B\cdot e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.

Proof: a) Assume $e\cdot cl_R = cl_R.e$ for ev $e\in I_R$. By Lemma 2.1, $e\cdot x = x\cdot e \quad \forall x\in cl_R$.

Since $I_R \subseteq cl_R$, we have $e\cdot e' = e'\cdot e$ for all e and e' of I_R. By Lemma 1.4, $I_R \subseteq C_R$.

b) This follows from the fact that $S.cl_R \subseteq cl_R$ and Lemma 2.1.

The proofs of c) and d) are similar to the proof of b) in Corollary 2.2.

Proposition 2.11

a) If $e\cdot P_R = P_R\cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.
b) If $e\cdot ST_R = ST_R\cdot e \quad \forall e\in I_R$ in a ring R in which $2x=0 \Rightarrow x=0$, then $I_R \subseteq C_R$.
c) If $e\cdot T_R = T_R\cdot e \quad \forall e\in I_R$ in a ring R in which $2x=0 \Rightarrow x=0$, then $I_R \subseteq C_R$.
d) If $e.\overline{J_R} = \overline{J_R}.e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.
e) If $e\cdot \overline{rad\ R} = \overline{rad\ R}\cdot e \quad \forall e\in I_R$, then $I_R \subseteq C_R$.

Proof: a) Suppose $e\cdot P_R = P_R\cdot e$, $\forall e\in I_R$. By Lemma 2.1, $e\cdot x = x\cdot e \quad \forall x\in P_R$.

Since $I_R \subseteq P_R$, we have $e\cdot e' = e'\cdot e$ for all e and e' in I_R. By 3) of Lemma 1.4, $I_R \subseteq C_R$.

b) Suppose $e\cdot ST_R = ST_R\cdot e$. This implies ,by Lemma 2.1, $ex = xe \quad \forall x\in ST_R$. Let $e'\in I_R$, then $(2e'-1)\in ST_R$, we have $e\cdot(2e'-1) = (2e'-1)\cdot e$. Therefore $2(ee'-e'e)=0$, i.e., $e\cdot e' = e'\cdot e \ \ \forall e, e'\in I_R$. Thus $I_R \subseteq C_R$ follows from 3) of Lemma 1.4.

c) This follows from the fact that $ST_R \subseteq T_R$.

d) We know that $I_R \subseteq \overline{J_R}\cup\{0\}$ [In fact $0\neq e\in I_R \Rightarrow (1-e)\in I_R \Rightarrow (1-e)\notin U_R \Rightarrow e\notin J_R$. This follows from proposition.5of [8]. So, $e\cdot\overline{J_R} = \overline{J_R}\cdot e$ which implies $e\cdot e' = e'\cdot e$ for all e and e' in I_R. By 3) of Lemma 1.4 $I_R \subseteq C_R$.

e) We have $I_R \subseteq \overline{rad\ R}\cup\{0\}$[In fact let $0\neq e\in I_R$. If $e\in rad\,R$. By Proposition 1 of [8], $e\in S_t N_R$. So $e\in N_R$, which implies $e=0$, which is a contradiction, so $e\in \overline{rad\ R}$]. Now, $e.\overline{rad\ R} = \overline{rad\ R}.e$ implies $e\cdot e' = e'\cdot e$, for all e and e' in I_R. By 3) of Lemma 1.4, $I_R \subseteq C_R$.

Remark 2.12: $e\cdot J_R = J_R\cdot e \quad \forall e\in I_R$ does not imply $I_R \subseteq C_R$, as shown in the next example.

Example 2.13 In the ring $M_{2\times2}(\Re) = R$, we have $J_R = \{0\}$. So $e \cdot J_R = J_R \cdot e \quad \forall e \in I_R$, but $I_R \not\subset C_R$ because $e = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \in I_R$ and $e \notin C_R$.

Similarly, the same example shows that $e \cdot rad\, R = rad\, R \cdot e \quad \forall e \in I_R$ does not imply $I_R \subseteq C_R$.

proposition 2.14

1) If $e.Z_R = Z_R.e$ for all $e \in I_R$, then $I_R \subseteq C_R$.
2) If $e.\overline{Z_R} = \overline{Z_R}.e \quad \forall e \in I_R$, then $I_R \subseteq C_R$.
3) If the elements of Z_R commute with themselves, then $I_R \subseteq C_R$.
4) If the elements of $\overline{Z_R}$ commute with themselves in a ring R in which $2x = 0 \Rightarrow x = 0$. Then $I_R \subseteq C_R$.
5) If the elements of Z_R commute with the elements of $\overline{Z_R}$ in a ring R in winch $2x = 0 \Rightarrow x = 0$, then $I_R \subseteq C_R$.

Proof: 1) If $\forall e \in I_R$. $e.Z_R = Z_R.e$. Then $e.z = z.e$ for all z $\in Z_R$. Since $SN_R \subseteq Z_R$, then $e.s = s.e \quad \forall s \in S$ and $\forall e \in I_R$. So $I_R \subseteq C_R$ follows from Lemma 2.3

2) If $e.\overline{Z_R} = \overline{Z_R}.e \quad \forall e \in I_R$, we have $e.x = x.e$ for all $x \in \overline{Z_R}$. Since $U_R \subseteq \overline{Z_R}$, $e.u = u.e \quad \forall u \in U_R$.

So $I_R \subseteq C_R$ follows from proposition 2.7.

3) Let $e, e' \in I_R$. Then $e, e' \in Z_R$. So $e.e' = e'.e$. By Lemma 1.4, $I_R \subseteq C_R$.

4) Let $e \in I_R$, $s \in SN_R$ .Then $(2e-1)$ and $(1-s) \in \overline{Z_R}$ .Now, $(2e-1)(1-s) = (1-s)(2e-1)$ $\Rightarrow 2(es - se) = 0 \Rightarrow es = se$. By Lemma 2.3 $I_R \subseteq C_R$.

5) Since $SN_R \subseteq Z_R$ and $(2e-1) \in \overline{Z_R} \ \forall e \in I_R$, we have $(2e-1)s = s(2e-1) \quad \forall s \in SN_R, \forall e \in I_R$ .Now, $2(es - se) = 0$ or $es = se$. By Lemma 2.3 $I_R \subseteq C_R$.

Corollary 2.15 1) If $e \cdot A.cl_R = A.cl_R \cdot e$, for all $e \in I_R$, then $I_R \subseteq C_R$.

2) If the elements of $A.cl_R$ commute with themselves, then $I_R \subseteq C_R$.

Proof: 1) Let $r \in \overline{Z_R}$. Then $r+1 \in A.cl_R$. Now, $e \cdot A.cl_R = A.cl_R \Rightarrow ex = xe \quad \forall x \in A.cl_R$ (by Lemma 2.1)

$\Rightarrow e(r+1) = (r+1)e \Rightarrow er = re$. So, by Proposition 2.14 $I_R \subseteq C_R$.

2)It is clear that $I_R \subseteq cl_R \subseteq A.cl_R$, so if $e, e' \in I_R$, then by hypotheses $ee' = e'e$, so, by 3) of Lemma 1.4 $I_R \subseteq C_R$.

Proposition 2.16 If R is a strongly clean ring, then R is a clean ring and $I_R \subseteq C_R$. However, the converse is not true**.**

Proof: Let $a \in R$. Since R is strongly clean, there exist $e \in I_R$ and $u \in U_R$ such that $a = eu = ue$. Let $e_1 = 1 - e$, $u_1 = a - e_1$ then $e_1 \in I_R$ and $u_1(u^{-1}e - e_1) = (u^{-1}e - e_1)u = 1$. So $u_1 \in U_R$ and we have $a = e_1 + u_1$, i.e., a is a clean element. Thus R is a clean ring.

The result $I_R \subseteq C_R$ follows from the fact that every strongly clean ring is a strongly regular ring (as we will see in Corollary 3.2).

The converse is not true because Z_4 is commutative clean ring but it is not regular, so it is not strongly clean.

Proposition 2.17 If R a fully generalized right primary right ring containing a non-idempotent maximal right ideal M, then $I_R = \{0,1\} \subseteq C_R$.

Proof: We will prove that M is the unique maximal right ideal of R which means R is a local ring. Therefore, $I_R = \{0,1\}$. Suppose there is another maximal right ideal M_1 of R. Then M_1M is a proper right ideal, and so it is a generalized right primary right ideal. Since $M_1M \subseteq M_1M$, we have either $M_1 \subseteq M_1M \subseteq M_1$ and hence $M_1 = M_1M$ or $\exists n \in N$ such that $M^n \subseteq M_1M \subseteq M_1$. Since M_1 is a semi prime right ideal (being a maximal right ideal), we get $M \subseteq M_1$. Hence $M = M_1$, because M is a maximal right ideal, but this contradicts the fact that $M_1 \neq M$. Thus $M_1 = M_1M$.

Next, $M^2 = MM = MM_1M = MM_1 = M$. This contradicts that M is not idempotent. Consequently, M is a unique maximal right ideal.

4. Rings With $I_R \subseteq C_R$

This section presents several properties of rings whose idempotents are central.

Lemma 3.1 [7] If $I_R \subseteq C_R$, then R is a regular ring if and only if R is a strongly regular ring.

Corollary 3.2 If $I_R \subseteq C_R$, then the following conditions are equivalent:

1) R is a regular ring.
2) R is a strongly regular ring.
3) R is a strongly clean ring

Proof: Follows directly from Lemma 1.7, Lemma 1.8, and Lemma 3.1.

Proposition 3.3 If $I_R \subseteq C_R$, then $regR = S.cl_R$.

Proof: Let $a \in S.cl_R$, then $\exists u \in U_R$ and $e \in I_R$ such that $a = ue$, so $ae = ue^2 = ue = a$.

Thus $u^{-1}a = e \Rightarrow au^{-1}a = ae = a \Rightarrow a \in regR$. Hence $S.cl_R \subseteq regR \ldots(1)$

On the other hand, let $a \in regR$, then $\exists\, b \in R$ such that $a = aba$ .Let $c = bab$, then

$a = aca$ and $c = cac$ .Since ac, ca are in $I_R \subseteq C_R$, we have $a = a^2c = ca^2$ and $c = c^2a = ac^2$, where $ac = ca^2c = ca$.

Now, $(1+a-ac)(1+c-ac) = 1 = (1+c-ac)(1+a-ac)$. Let $u = 1+a-ac$ and $e = ac$, then $u \in U_R, e \in I_R$ and we have $a = ue = eu$. So $a \in S \cdot cl_R$. Hence $regR \subseteq S \cdot cl_R \ldots(2)$.

By (1) and (2) we have $regR = S \cdot cl_R$.

It is mentioned in [9] without proof that every periodic ring is clean ring

We will add the condition $I_R \subseteq C_R$ and give easy proof for this result.

Proposition 3.4 Suppose $I_R \subseteq C_R$. If R periodic ring, then R is a clean ring.

Proof: Let $a \in R$. Since a is periodic, there are $m, n \in N$ with $m > n$ such that $a^m = a^n$, we have $a^{n(m-n)} \in I_R$. Let $n(m-n) = s$ and $a^s = e \in I_R \subseteq C_R$, then $e = (2e-1)+(1-e) = u_1 + e_1$.

i.e., $a^s = u_1 + e_1$, where $u_1 = (2e-1) \in U_R$ and $e_1 = (1-e) \in I_R$. Thus a^s is a clean element.

Now, $a^s - e_1^s = (a-e_1)(a^{s-1} + e_1 a^{s-2} + \ldots + e_1 a + e_1)$.Put $x = a^{s-1} + e_1 a^{s-2} + \ldots + e_1 a + e_1$.

Then $(a-e_1)x = a^s - e_1^s = a^s - e_1 = u_1$. Therefore $u_1^{-1}(a-e_1)x = 1$.

In the same way, $x(a-e_1)u_1^{-1} = 1$. Thus x is invertible and $x^{-1} = (a-e_1)u_1^{-1} = u_1^{-1}(a-e_1)$,

From the equation $(a-e_1)x = u_1$ it follows that $a - e_1 = u_1 x^{-1} = u_2$ and $u_2 \in U_R$.

Hence $a = u_2 + e_1$, i.e., a is a clean element. Therefore R is clean ring.

Although the following Corollary might be found somewhere else, we add it to integrate our work.

Corollary 3.6 Suppose $I_R \subseteq C_R$. If R is a $\pi-$regular ring, then R is a clean ring.

Proof: Let $a \in R$. Then there exist $n \in N$ such that $a^n = a^n b a^n; b \in R$. Let $f = a^n b$ then $f \in I_R \subseteq C_R$. So $f - 1 \in C_R$. Let $u = a^n + (f-1)$.Then: $u \cdot [bf + (f-1)] = 1 = [bf + (f-1)]$.u , that is u is a unit.

We have $a^n = u + (1-f)$ with $(1-f) \in I_R$ and $u \in U_R$ .Thus $a^n = u + e$ where u $\in U_R$ and e$\in I_R$, i.e., a^n is a clean element, we now repeat the same technique of the proof of proposition 3.4 to get a is clean element, and R is a clean ring.

Proposition 3.7 The following conditions are equivalent:

1) R Is a regular ring and $I_R \subseteq C_R$.
2) R Is a fully right semi prime ring and semi commutative ring.

Proof: $1 \Rightarrow 2$: Since R is regular, each right ideal of R is idempotent. This makes R a fully right semi prime[3]. On the other hand, let $a, r \in R$, since R is regular, $\exists\, b \in R; a = aba$ such that $ba \in I_R \subseteq C_R$. So, $ar = a(ba)r = ar(ba) = r'a, \quad ba \in I_R \subseteq C_R$.

In the same way, $ra = ar''$. So R is semi commutative.

$2 \Rightarrow 1$: Let $a \in R$, the aR is a right ideal in R. Since R is a fully semi prime ring, aR is an idempotent right ideal [3]. Since R is semi commutative, $aR = Ra$. Hence $a \in aR = aR \cdot aR \subseteq aRa$. Thus $a = ara$. This means that a is a regular element. Therefore R is a regular ring.

On other hand, by Lemma 1.4, $I_R \subseteq C_R$.

Corollary 3.8 If $I_R \subseteq C_R$, then the following conditions are equivalent:

1) R Is a regular ring.
2) $a \in (aR \bigcap Ra)^2 \quad \forall a \in R$ . .

Proof: $1 \Rightarrow 2$ as follow if R is regular and $I_R \subseteq C_R$, R is semi commutative (by Proposition 3.7). So $aR = Ra$, so $aR \bigcap Ra = aR$. Since R is regular, we obtain $(aR)^2 = aR$ and $(aR \bigcap Ra)^2 = aR$. Since $a \in aR$ .we have $a \in (aR \bigcap Ra)^2$.

$2 \Rightarrow 1$: Let $a \in R$ then by 2), $a \in (Ra \bigcap aR)^2 = (Ra \bigcap aR)(Ra \bigcap aR) \subseteq aR \cdot Ra \subseteq aRa$.

So, $\exists b \in R$ such that $a = aba$. Hence a is a regular element and R is a regular ring.

Lemma 3.9 Suppose $I_R \subseteq C_R$. If R is a generalized right primary right ring, then $I_R = \{0,1\}$

Proof: Let $e \in I_R$. Since $I_R \subseteq C_R$, eR .(1-e)R = Re(1-e)R $\subseteq \{0\}$. Since $\{0\}$ is a generalized right primary right ideal, either $e \in eR \subseteq \{0\}$ and hence $e = 0$ or $\exists n \in N$ such that

$$(1-e)(1-e)^n \in [(1-e)R]^n \subseteq \{0\}$$, which yields $e = 1$. Thus $I_R = \{0,1\}$.

Corollary 3.10 Suppose $I_R \subseteq C_R$. If R is a right prime ring, $I_R = \{0,1\}$.

Proof: Every right prime ideal is a generalized right primary right ring.

Corollary 3.11 Suppose $I_R \subseteq C_R$. If I right prime (or R is a generalized right primary right ring) and right artinian ring, then R is a local ring.

Proof: It is a well known that if R is artinian and $I_R = \{0, 1\}$, then R is a local ring.

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