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EMPIRICAL BAYES ESTIMATES OF RAYLEIGH DISTRIBUTION WITH EWMEL AND LOGARITHMIC LOSS FUNCTIONS FOR CENSORED SAMPLES

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ABSTRACT: In this paper, empirical Bayes estimates of reliability performances are derived when the data are progressively Type II censored from a Rayleigh distribution. These estimates are derived under exponentially weighted minimum expected loss (EWMEL) and logarithmic loss functions, and compared with the corresponding maximum likelihood estimates in terms of absolute bias and estimated risk. A real data set is presented to illustrate the proposed estimation method, and a Monte Carlo simulation study is carried out to investigate the accuracy of derived estimates. The study shows that the empirical Bayesian estimation outperforms the maximum likelihood estimation.

1. Introduction

Rayleigh distribution is widely used to model events which occur in different fields such as medicine, social and natural sciences, communication engineering, reliability and life testing, and applied statistics. Lord Rayleigh [22] invented this distribution from the amplitude of sound resulting from many important sources. Polovko [21] demonstrated the importance of this distribution in communication engineering and electro vacuum devices. Siddique [27] has used this distribution as a radio wave power distribution. Bhattacharya and Tyagi [7] applied this distribution in some clinical studies dealing with cancer patients.

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The probability density and cumulative distribution functions of Rayleigh distribution are given, respectively, by

$$f(x \mid \theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}, \quad x > 0, \quad \theta > 0, \tag{1.1}$$

and

$$F(x \mid \theta) = 1 - e^{-\frac{x^2}{2\theta^2}}. \tag{1.2}$$

Moreover, the reliability and failure rate functions are given, respectively, by

$$R(t) = e^{-\frac{t^2}{2\theta^2}} \text{ and } \lambda(t) = \frac{t}{\theta^2}, \quad t > 0. \tag{1.3}$$

Many authors have developed statistical inference procedures for Rayleigh distribution. For example, among others, Howlader and Hossian [14] derived the Bayes estimates for scale parameter and reliability function of the Rayleigh distribution in the case of Type II censored samples. Wu et al. [28] and Lee et al. [16] have derived the maximum likelihood and Bayes estimates of reliability performances of the Rayleigh lifetime model under the squared error loss function in the case of progressive Type II censoring.

The empirical Bayesian approach has become quite popular in the theory and practice of statistics in the last three decades. This approach was first formulated by Robbins [24], and has been used rather extensively by several authors. For example, among others, Ali Mousa [1] has obtained the empirical Bayes estimates for the Burr type XII model based on Type II censored data. Asgharzadeh and Valiohi [4] have studied the problems of estimation and prediction for the proportional hazards family under progressive Type II censoring through empirical Bayesian approach. Rezaeian and Asgharzadeh [23] have obtained the Bayes and empirical Bayes estimates of scale parameter of the Gamma distribution under balanced loss functions. This approach has been described extensively by many authors ([9], [17], [19], [10]).

However, up to now, empirical Bayes estimates of reliability performances of the Rayleigh model based on EWMEL and logarithmic loss functions were not addressed under progressively Type II censoring. The main aim of a paper is to obtain empirical Bayes estimates of the scale parameter, reliability function, and failure rate function of the Rayleigh model based on progressively Type II censored samples, and to compare them with the corresponding maximum likelihood estimates in terms of bias and posterior risk.

The rest of the paper is organized as follows. In Section 2, a progressive Type II censoring scheme without replacement is discussed, and maximum likelihood estimates of reliability performances for the Rayleigh model under this censoring are stated. In Section 3, Bayesian estimation of reliability performances under EWMEL and logarithmic loss functions is considered. In Section 4, empirical Bayes estimates of reliability performances are derived where the hyper-parameter is estimated using maximum likelihood approach. In Section 5, a real data provided by Lawness [15] is analyzed to illustrate the proposed estimation method. Finally, in Section 6, a Monte-Carlo simulation study is carried out to compare the performance of empirical Bayes estimates with the corresponding maximum likelihood estimates. The paper concludes in Section 7.

2. PROGRESSIVE TYPE II CENSORING SCHEME

Censoring is used in life testing to save time and cost. The most popular censoring schemes, among the various types of censoring schemes used in lifetime analysis, are Type I and Type II censoring schemes. These types of censoring cannot allow removal of units at points other than the terminal point of an experiment. However, this allowance may be desirable, as in the case of accidental breakage of test units where the loss of units at points other than the terminal point may be unavoidable. This leads us into the area of more general censoring scheme called progressive Type II censoring scheme. This censoring scheme is also useful in many practical situations where budget constraints are in place or there is demand for rapid testing. Statistical inference problems for various lifetime distributions under progressive Type II censoring have been discussed by several authors ([2], [3], [11], [12], [20]). The interested readers may refer to the book by Balakrishnan and Aggarwala ([6], Chapter 1) for additional discussions on need for progressive censoring.

A progressive Type II censoring scheme without replacement can be described as follows. Consider an experiment in which n independent and identical units are placed on a life test at the beginning time, and the failure times of first m $(1 \le m < n)$ units are recorded. That is, instead of continuing until all n units have failed, the life test is terminated at the time of m^{th} failure. At the time of each failure occurring prior to the termination point, one or more surviving units are removed from the life test, that is, r_1 of the $(n-1)$ surviving units are withdrawn at the time of the first failure, r_2 of the $(n-2-r_1)$ surviving units are withdrawn at the time of the second failure, and so on. Finally, at the time of the m^{th} failure, the life test is terminated and all the remaining $r_m \left(= n - m - \sum_{i=1}^{m-1} r_i\right)$ surviving units are withdrawn. When $r_1 = r_2 = \dots = r_m = 0$ and $n = m$, the progressive Type II censoring scheme reduces to complete sampling scheme; and when $r_1 = r_2 = \dots = r_{m-1} = 0$ and $r_m = n - m$, this scheme reduces to conventional Type II censoring scheme.

Let $x_{(1)} \le x_{(2)} \le \dots \le x_{(m)}$ be the failure times of completely observed units to fail; $r_1, r_2, \dots, r_m$ be the number of units withdrawn at these failure times; and $\underline{x} = \left(x_{(1)}, x_{(2)}, \dots, x_{(m)}\right)$ be the progressively Type II censored sample of the life test on units whose lifetimes have a Rayleigh distribution with the probability density function (1.1). The likelihood function based on $\underline{x}$ is (see Balakrishnan and Aggarwala, [6])

$$L(\underline{x} \mid \theta) = A \prod_{i=1}^{m} f\left(x_{(i)} \mid \theta\right)\left[1 - F\left(x_{(i)} \mid \theta\right)\right]^{r_i}, \qquad (2.1)$$

where

$$A = n\left(n-1-r_1\right)\left(n-2-r_1-r_2\right) \, \dots \, \left(n-m+1-\sum_{i=1}^{m-1} r_i\right).$$

From (1.1), (1.2), and (2.1), the likelihood function L is found to be

$$L(\underline{x} \mid \theta) = \frac{A \prod_{i=1}^{m} x_{(i)}}{\theta^{2m}} e^{-\frac{1}{2\theta^2} \sum_{i=1}^{m} x_{(i)}{}^2 (1+r_i)}. \qquad (2.2)$$

The maximum likelihood estimates of the scale parameter θ, reliability function $R(t)$, and failure rate function $\lambda(t)$ of the Rayleigh model based on $\underline{x}$ (Refer Wu et al. [28]) are given, respectively, by

$$\hat{\theta}_{MLE} = \sqrt{\frac{\sum_{i=1}^{m} x_{(i)}^2 (1+r_i)}{2m}}, \qquad (2.3)$$

$$\hat{R}(t)_{MLE} = e^{-\frac{t^2}{2\hat{\theta}_{MLE}^2}}, \qquad (2.4)$$

$$\hat{\lambda}(t)_{MLE} = \frac{t}{\hat{\theta}_{MLE}^2}. \qquad (2.5)$$

3. BAYES ESTIMATES OF RELIABILITY PERFORMANCES

In this section, we consider Bayesian estimation of the scale parameter θ, reliability function $R(t)$, and failure rate function $\lambda(t)$ of the Rayleigh model. Suppose that the unknown scale parameter θ is the realization of a random variable, which has an inverted gamma prior with the probability density function

$$\pi(\theta \mid b) = \frac{\theta^{-2b-1} e^{-\frac{1}{2\theta^2}}}{\Gamma(b)\, 2^{b-1}}, \quad b > 0, \qquad (3.1)$$

where b is the unknown hyper-parameter chosen to reflect prior beliefs on θ.

This prior distribution has advantages over many other distributions because of its analytical tractability, richness, and easy interpretability. The joint probability density function of θ and $\underline{x}$ is given by

$$g(\underline{x}, \theta) = L(\underline{x} \mid \theta)\, \pi(\theta \mid b) = \frac{A \prod_{i=1}^{m} x_{(i)}}{\Gamma(b)\, 2^{b-1}} e^{-\frac{w}{2\theta^2}} \theta^{-2(b+m)-1},$$

where

$$w = \sum_{i=1}^{m} x_{(i)}^{2}(1 + r_i) + 1.$$

From Bayes' theorem, the posterior probability distribution of θ can be written as

$$\pi^{*}(\theta \mid \underline{x}) = \frac{g(\underline{x}, \theta)}{\int_0^\infty g(\underline{x}, \theta)\, d\theta} = \frac{w^{b+m}}{\Gamma(b+m)\, 2^{b+m-1}} e^{-\frac{w}{2\theta^2}} \theta^{-2(b+m)-1}. \qquad (3.2)$$

In order to derive Bayes estimates, one must have to specify a loss function, which represents a penalty associated with each of the possible estimates. The loss function is a non - negative function of the distance between estimate and true value. To control the amount of variability, the most widely used loss function is a quadratic loss function in the form $L_1(\hat{\phi}, \phi) = k(\hat{\phi} - \phi)^2$, where $\hat{\phi}$ is an estimate of ϕ. This loss function is symmetrical and gives equal importance to the losses due to overestimation and underestimation of equal magnitude. If k is a function of ϕ, the loss function is termed as weighted quadratic loss function.

The EWMEL function can be obtained with the choice of $k(\phi) = \phi^{-2} e^{-\frac{c}{2\phi^2}}$.

The Bayes estimate of ϕ under an EWMEL function, denoted by $\hat{\phi}_{BE}$, is the value of $\hat{\phi}$ that minimizes the posterior expectation of the loss function. It is

$$\hat{\phi}_{BE} = \frac{E^{\pi^*}(k(\phi)\phi \mid \underline{x})}{E^{\pi^*}(k(\phi) \mid \underline{x})}.$$

For $c = 0$, this loss function reduces to minimum expected loss (MEL) function with $k(\phi) = \phi^{-2}$, which was proposed by Tummala and Sathe [25].

Another loss function in popular use is a logarithmic loss function that places a small weight on estimates whose ratios to the true value are close to one, and proportionately more weight on estimates whose ratios to the true value are significantly different from one.

This loss function was introduced by Brown [8], and can be expressed as

$$L_2\left(\hat{\phi}, \phi\right) = \left(\ln \frac{\hat{\phi}}{\phi}\right)^2 = \left(\ln \hat{\phi} - \ln \phi\right)^2.$$

The Bayes estimate of ϕ under the logarithmic loss function, denoted by $\hat{\phi}_{BL}$, is the value of $\hat{\phi}$ that minimizes the posterior expectation of the loss function. It is

$$\hat{\phi}_{BL} = \exp\left[E^{\pi^*}\left(\ln \phi \mid \underline{x}\right)\right].$$

Shah and Patel [26] derived the Bayes estimates of reliability performances of the Rayleigh distribution under the EWMEL function based on multiply Type II censored data. Asgharzadeh and Valiollahi [3] obtained the empirical Bayes estimates of unknown parameter and reliability function of Burr distribution under absolute error and logarithmic loss functions based on progressively Type II censored data.

3.1. Bayes Estimates of Reliability Performances under the EWMEL Function

The Bayes estimate of scale parameter θ under the EWMEL function is

$$\hat{\theta}_{BE} = \frac{E^{\pi^*}\left(k(\theta)\theta \mid \underline{x}\right)}{E^{\pi^*}\left(k(\theta) \mid \underline{x}\right)} = \frac{\int_0^\infty \theta^{-1} e^{-\frac{c}{2\theta^2}} \pi^*\left(\theta \mid \underline{x}\right) d\theta}{\int_0^\infty \theta^{-2} e^{-\frac{c}{2\theta^2}} \pi^*\left(\theta \mid \underline{x}\right) d\theta}$$

$$= \frac{\int_0^\infty e^{-\frac{c+w}{2\theta^2}} \theta^{-2(b+m)-2} d\theta}{\int_0^\infty e^{-\frac{c+w}{2\theta^2}} \theta^{-2(b+m)-3} d\theta}$$

$$= \frac{\Gamma\left(b+m+\frac{1}{2}\right)\sqrt{\frac{c+w}{2}}}{\Gamma(b+m+1)}. \tag{3.3}$$

Moreover, the Bayes estimates of reliability function $R(t)$ and failure rate function $\lambda(t)$ under the EWMEL function at mission time t are, respectively,

$$\hat{R}(t)_{BE}=\frac{E^{\pi^*}\left(k(R(t))\,R(t)\,|\,\underline{x}\right)}{E^{\pi^*}\left(k(R(t))\,|\,\underline{x}\right)}=\frac{\int_0^\infty e^{\frac{t^2}{2\theta^2}}\;e^{-\frac{c\,e^{\frac{t^2}{\theta^2}}}{2}}\;\pi^*(\theta\,|\,\underline{x})\,d\theta}{\int_0^\infty e^{\frac{t^2}{\theta^2}}\;e^{-\frac{c\,e^{\frac{t^2}{\theta^2}}}{2}}\;\pi^*(\theta\,|\,\underline{x})\,d\theta}$$

$$=\frac{\sum\limits_{s=0}^{\infty}\frac{(-c)^s}{2^s\,s\,!}\int_0^\infty e^{\frac{(2s+1)t^2}{2\theta^2}}\;e^{-\frac{w}{2\theta^2}}\;\theta^{-2(b+m)-1}\;d\theta}{\sum\limits_{s=0}^{\infty}\frac{(-c)^s}{2^s\,s\,!}\int_0^\infty e^{\frac{(s+1)t^2}{2\theta^2}}\;e^{-\frac{w}{2\theta^2}}\;\theta^{-2(b+m)-1}\;d\theta}$$

$$=\frac{\sum\limits_{s=0}^{\infty}\sum\limits_{p=0}^{\infty}\frac{(-c)^s\left[(2s+1)t^2\right]^p}{2^{s+p}\,s\,!\,p\,!}\int_0^\infty e^{-\frac{w}{2\theta^2}}\;\theta^{-2(b+m+p)-1}\;d\theta}{\sum\limits_{s=0}^{\infty}\sum\limits_{p=0}^{\infty}\frac{(-c)^s\left[(s+1)t^2\right]^p}{2^{s}\,s\,!\,p\,!}\int_0^\infty e^{-\frac{w}{2\theta^2}}\;\theta^{-2(b+m+p)-1}\;d\theta}$$

$$=\frac{\sum\limits_{s=0}^{\infty}\sum\limits_{p=0}^{\infty}\frac{(-c)^s\left[(2s+1)t^2\right]^p\,\Gamma(b+m+p)}{2^s\,s\,!\,p\,!\,w^p}}{\sum\limits_{s=0}^{\infty}\sum\limits_{p=0}^{\infty}\frac{(-c)^s\left[2(s+1)t^2\right]^p\,\Gamma(b+m+p)}{2^s\,s\,!\,p\,!\,w^p}}, \qquad (3.4)$$

and

$$\hat{\lambda}(t)_{BE}=\frac{E^{\pi^*}\left(k(\lambda(t))\,\lambda(t)\,|\,\underline{x}\right)}{E^{\pi^*}\left(k(\lambda(t))\,|\,\underline{x}\right)}=\frac{\int_0^\infty \frac{\theta^2}{t}\,e^{-\frac{c\,\theta^4}{2t^2}}\;\pi^*(\theta\,|\,\underline{x})\,d\theta}{\int_0^\infty \frac{\theta^4}{t^2}\,e^{-\frac{c\,\theta^4}{2t^2}}\;\pi^*(\theta\,|\,\underline{x})\,d\theta}$$

$$=\frac{t\sum\limits_{s=0}^{\infty}\frac{\left(-c/2t^2\right)^s}{s!}\int_0^\infty e^{-\frac{w}{2\theta^2}}\;\theta^{-2(b+m-2s-1)-1}\,d\theta}{\sum\limits_{s=0}^{\infty}\frac{\left(-c/2t^2\right)^s}{s!}\int_0^\infty e^{-\frac{w}{2\theta^2}}\;\theta^{-2(b+m-2s-2)-1}\,d\theta}$$

$$= \frac{t \sum_{s=0}^{\infty} \frac{\left(-c/2t^2\right)^s \Gamma(b+m-2s-1)\ 2^{b+m-2s-2}}{s!\ w^{b+m-2s-1}}}{\sum_{s=0}^{\infty} \frac{\left(-c/2t^2\right)^s \Gamma(b+m-2s-2)\ 2^{b+m-2s-3}}{s!\ w^{b+m-2s-2}}}$$

$$= \frac{2t \sum_{s=0}^{\infty} \frac{\left(-cw^2/8t^2\right)^s \Gamma(b+m-2s-1)}{s!}}{w \sum_{s=0}^{\infty} \frac{\left(-cw^2/8t^2\right)^s \Gamma(b+m-2s-2)}{s!}}. \tag{3.5}$$

3.2. BAYES ESTIMATES OF RELIABILITY PERFORMANCES UNDER THE LOGARITHMIC LOSS FUNCTION

The Bayes estimate of scale parameter θ under the logarithmic loss function is

$$\hat{\theta}_{BL} = \exp\left[E^{\pi^*}(\ln\theta \mid \underline{x})\right]$$

$$= \exp\left[\frac{w^{b+m}}{\Gamma(b+m)\,2^{b+m-1}} \int_0^\infty \ln\theta\, e^{-\frac{w}{2\theta^2}}\ \theta^{-2(b+m)-1}\, d\theta\right]$$

$$= \exp\left\{\frac{-1}{2}\left\{\ln\left(\frac{2(b+m)}{w}\right) - \sum_{k=0}^{\infty}\left[\frac{1}{b+m+k} - \ln\left(1+\frac{1}{b+m+k}\right)\right]\right\}\right\}. \tag{3.6}$$

(Refer Gradshteyn and Ryzhik [13], pp. 893).

Moreover, the Bayes estimates of reliability function $R(t)$ and failure rate function $\lambda(t)$ under the logarithmic loss function at mission time t are, respectively,

$$\hat{R}(t)_{BL} = \exp\left[E^{\pi^*}(\ln R(t) \mid \underline{x})\right]$$

$$= \exp\left[\frac{-w^{b+m}\ t^2}{\Gamma(b+m)\,2^{b+m}} \int_0^\infty e^{-\frac{w}{2\theta^2}}\ \theta^{-2(b+m+1)-1}\, d\theta\right]$$

$$= e^{-\frac{(b+m)\ t^2}{w}}, \tag{3.7}$$

and

$$\hat{\lambda}(t)_{BL} = \exp\left[E^{\pi^*}(\ln \lambda(t) \mid \underline{x})\right]$$

$$= \exp\left[\frac{w^{b+m}}{\Gamma(b+m)\,2^{b+m-1}} \int_0^\infty (\ln t - 2\ln\theta)\, e^{-\frac{w}{2\theta^2}}\, \theta^{-2(b+m)-1}\, d\theta\right]$$

$$= \exp\left\{\ln t + \ln\left(\frac{2(b+m)}{w}\right) - \sum_{k=0}^{\infty}\left[\frac{1}{b+m+k} - \ln\left(1+\frac{1}{b+m+k}\right)\right]\right\} \quad (3.8)$$

4. EMPIRICAL BAYES ESTIMATES OF RELIABILITY PERFORMANCES

The Bayes estimates obtained in previous Section are seen to depend on the hyper-parameter b of prior distribution (3.1). As the hyper-parameter b is unknown, we may use empirical Bayesian approach to get its estimate. In empirical Bayesian approach, we begin with the Bayes model

$$x_{(i)} \mid \theta \sim f(x \mid \theta), \quad i = 1, 2, \ldots, m,$$

$$\theta \mid b \sim \pi(\theta \mid b).$$

The marginal distribution of $\underline{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(m)})$, say $m(\underline{x} \mid b)$, can be obtained by

$$m(\underline{x} \mid b) = \int_0^\infty L(\underline{x} \mid \theta)\, \pi(\theta \mid b)\, d\theta. \quad (4.1)$$

It follows, from (2.2), (3.1), and (4.1), that the marginal distribution of $\underline{x}$ is

$$m(\underline{x} \mid b) = \frac{A \prod_{i=1}^{m} x_{(i)}}{\Gamma(b)\,2^{b-1}} \int_0^\infty e^{-\frac{w}{2\theta^2}}\, \theta^{-2(b+m)-1}\, d\theta = \frac{A \prod_{i=1}^{m} x_{(i)}\, \Gamma(b+m)\, 2^m}{\Gamma(b)\, w^{b+m}}. \quad (4.2)$$

Based on $m(\underline{x} \mid b)$, we can obtain an estimate $\hat{b}$ of b. It is most common to take $\hat{b}$ to be the maximum likelihood estimate of b, but it is not essential.

The natural logarithm of (4.2) is

$$\ln m(\underline{x}\,|\,b)=\ln A+\ln\prod_{i=1}^{m}x_{(i)}+\ln\Gamma(b+m)+m\ln 2-\ln\Gamma(b)-(b+m)\ln w. \quad (4.3)$$

Using the approximation of $\ln\Gamma(b)$ that works good even for small values of b (see Gradshteyn and Ryzhik, [13], pp. 888, 8.341(2)) in (4.3), we differentiate it with respect to b. Equalizing the obtained expression to zero, the likelihood equation is found to be

$$\left[\ln(b+m)-\frac{1}{2(b+m)}\right]-\left[\ln b-\frac{1}{2b}\right]-\ln w=0,$$

$$\text{i.e.,}\quad \ln\left(\frac{b+m}{bw}\right)+\frac{m}{2b(b+m)}=0. \qquad (4.4)$$

The equation (4.4) has no closed form solution. Therefore, an estimate $\hat{b}$ for b is obtained by solving the equation via numerical method. According to Lehmann and Casella [17], the empirical Bayes estimates of reliability performance of the Rayleigh model can be obtained by substituting $\hat{b}$ for b in the Bayes estimates. By substituting $\hat{b}$ for b in (3.3), (3.4) and (3.5), the empirical Bayes estimates of θ, $R(t)$ and $\lambda(t)$ under the EWMEL function can be obtained, respectively, as

$$\hat{\theta}_{EBE}=\frac{\Gamma\left(\hat{b}+m+\frac{1}{2}\right)\sqrt{\frac{c+w}{2}}}{\Gamma\left(\hat{b}+m+1\right)}, \qquad (4.5)$$

$$\hat{R}(t)_{EBE}==\frac{\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-c)^s\left[(2s+1)t^2\right]^p\,\Gamma\left(\hat{b}+m+p\right)}{2^s\,s!\,p!\,w^p}}{\sum_{s=0}^{\infty}\sum_{p=0}^{\infty}\frac{(-c)^s\left[2(s+1)t^2\right]^p\,\Gamma\left(\hat{b}+m+p\right)}{2^s\,s!\,p!\,w^p}}, \qquad (4.6)$$

$$\hat{\lambda}(t)_{EBE}=\frac{2t\sum_{s=0}^{\infty}\frac{\left(-cw^2/8t^2\right)^s\,\Gamma\left(\hat{b}+m-2s-1\right)}{s!}}{w\sum_{s=0}^{\infty}\frac{\left(-cw^2/8t^2\right)^s\,\Gamma\left(\hat{b}+m-2s-2\right)}{s!}}. \qquad (4.7)$$

Similarly, the empirical Bayes estimates of θ, $R(t)$ and $\lambda(t)$ under the logarithmic loss function can be obtained, respectively, as

$$\hat{\theta}_{EBL} = \exp\left\{\frac{-1}{2}\left\{\ln\left(\frac{2(\hat{b}+m)}{w}\right) - \sum_{k=0}^{\infty}\left[\frac{1}{\hat{b}+m+k} - \ln\left(1+\frac{1}{\hat{b}+m+k}\right)\right]\right\}\right\}, \tag{4.8}$$

$$\hat{R}(t)_{EBL} = e^{-\frac{(\hat{b}+m)\,t^2}{w}}, \tag{4.9}$$

$$\hat{\lambda}(t)_{EBL} = \exp\left\{\ln t + \ln\left(\frac{2(\hat{b}+m)}{w}\right) - \sum_{k=0}^{\infty}\left[\frac{1}{\hat{b}+m+k} - \ln\left(1+\frac{1}{\hat{b}+m+k}\right)\right]\right\}. \tag{4.10}$$

5. NUMERICAL EXAMPLE (REAL DATA)

In this section, a real data set reported in Lawless ([15], pp. 228) is analyzed to illustrate the proposed estimation method described in the preceding Section. Leiblein and Zelen [18] originally discussed this data set during the endurance test of 23 deep groove ball bearings. The failure times (in thousands of million revolutions) were:

0.01788	0.02892	0.03300	0.04152	0.04212	0.04560	0.04848
0.05184	0.05196	0.05412	0.05556	0.06780	0.06864	0.06864
0.06888	0.08412	0.09312	0.09864	0.10512	0.10584	0.12792
0.12804	0.17340					

We have checked the validity of the Rayleigh model based on the estimated value (moment estimate) of parameter $\theta = 0.0525$, using the Kolmogorov-Smirnov (KS) test. It is observed that the value of KS Statistic is 0.09463 with the corresponding tabulated value 0.275. This indicates that the Rayleigh model is adequate for the given data. For $\theta = 0.0525$, the reliability $R(t)$ and failure rate $\lambda(t)$ (at t = 0.04) are respectively 0.74807 and 14.51247.

As a numerical illustration, we have generated an artificial progressive Type II censored sample of size $m = 12$ and conventional Type II censored sample of size $m = 5$ from the given data set. For these two cases, the failure times along with the applied censoring schemes are reported in Tables 1 and 3. The maximum likelihood estimates and empirical Bayes estimates under the EWMEL, MEL, and logarithmic loss functions were computed for both the censored samples and reported in Tables 2 and 4.

Table 1. Progressively Type II censored sample

i	1	2	3	4	5	6
$x_{(i)}$	0.01788	0.02892	0.03300	0.04560	0.05184	0.05196
r_i	0	1	2	0	2	0
i	7	8	9	10	11	12
$x_{(i)}$	0.05556	0.06864	0.09312	0.10512	0.10584	0.12792
r_i	3	0	0	1	0	2

Table 2. Maximum likelihood and empirical Bayes estimates of θ, $R(t)$ and $\lambda(t)$ for progressively Type II Censored sample

Parameter	MLE	EWMEL function		MEL function	logarithmic Loss Function
		$c = -0.5$	$c = 0.5$	$c = 0$	
θ	0.07252	0.05389	0.08684	0.07227	0.07237
$R(t)$	0.85889	0.85789	0.85814	0.85802	0.85859
$\lambda(t)$	7.60550	7.49753	7.49883	7.49818	7.63694

Table 3. Conventional Type II censored sample

i	1	2	3	4	5
$x_{(i)}$	0.01788	0.02892	0.03300	0.04152	0.04212
r_i	0	0	0	0	18

Table 4. Maximum likelihood and empirical Bayes estimates of θ, $R(t)$ and $\lambda(t)$ for conventional Type II censored sample

Parameter	MLE	EWMEL function		MEL function	logarithmic Loss Function
		$c=-0.5$	$c=0.5$	$c=0$	
θ	0.06138	0.04407	0.07453	0.06122	0.06128
$R(t)$	0.80869	0.80751	0.80792	0.80772	0.80852
$\lambda(t)$	10.61662	10.49767	10.49838	10.49803	10.64937

From Tables 2 and 4, it is observed that (i) The empirical Bayes estimate of scale parameter θ under the EWMEL function is sensitive to the value of shape parameter c whereas of reliability function $R(t)$ and failure rate function $\lambda(t)$ are not. (ii) When the negative (positive) value of shape parameter c tends to zero from the left (right) side, the empirical Bayes estimate of scale parameter θ the EWMEL function get very closer to its maximum likelihood estimate. (iii) The empirical Bayes estimate of scale parameter under the EWMEL function get very closer to the corresponding maximum likelihood estimates.

6. SIMULATION STUDY

In this section, an extensive Monte Carlo simulation study is conducted to compare the performance of proposed Bayes estimates with the maximum likelihood estimates in terms of absolute bias and estimated risk for different sample sizes, effective sample sizes, shape parameter values, and censoring schemes with five withdrawal patterns.

The maximum likelihood and empirical Bayes estimates of reliability performances are computed according to the following steps:

1. For the given value of prior parameter $b = 50$, we have generated $\theta = 0.09216$ from (3.1) and then by using this generated value of θ, $R(t) = 0.91012$ and $\lambda(t) = 4.70907$ (at $t = 0.04$) are obtained from (1.3).
2. For a particular sample size (n), effective sample size (m), and censoring scheme $\underline{r} = (r_1, r_2, \ldots, r_m)$, we have generated a progressive Type II censored sample $\underline{U} = (U_1, U_2, \ldots, U_m)$ from the uniform distribution, and then $\underline{x} = (x_{(1)}, x_{(2)}, \ldots, x_{(m)})$ from the Rayleigh distribution according to the algorithm given in Balakrishnan and Sandhu [5], where $x_{(i)} = \sqrt{-2\theta^2 \ln(1 - U_i)}$, $i = 1, 2, 3, \ldots, m$.
3. For different values of c, we have computed the maximum likelihood and empirical Bayes estimates of θ, $R(t)$, and $\lambda(t)$ (at $t = 0.04$) using the results outlined in Sections 2 and 4.

As one data set does not help to clarify the performance of an estimate, the average estimates, absolute biases, and estimated risks of maximum likelihood and empirical Bayes estimates were computed on the basis of 1000 simulated data sets according to the following formulae:

Average estimate = $\dfrac{\sum_{i=1}^{1000} \hat{q}_i}{1000}$,

Absolute bias of an estimate = $|\text{Average estimate} - q|$,

Estimated risk = $\dfrac{\sum_{i=1}^{1000} (\hat{q}_i - q)^2}{1000}$.

Here, $\hat{q}$ is an estimate of q. All the calculations have been done through computer software Microsoft Visual Studio 2008.

Table 5 represents the different cases of sample sizes, effective sample sizes, and censoring schemes. The maximum likelihood and empirical Bayes estimates of reliability performances, and the corresponding estimated risks are reported, respectively, in Tables 6 - 8.

Table 5. Progressively Type II Censoring Schemes (C.S.) applied in the simulation study

n	*m*	C.S. No.	$\underline{r} = (r_1, r_2, \ldots, r_m)$
20	16	[1]	1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1
		[2]	2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
		[3]	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 2, 2
		[4]	4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
		[5]	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 4
	8	[6]	3, 3, 0, 0, 0, 0, 3, 3
		[7]	4, 4, 4, 0, 0, 0, 0, 0
		[8]	0, 0, 0, 0, 0, 4, 4, 4
		[9]	12, 0, 0, 0, 0, 0, 0, 0
		[10]	0, 0, 0, 0, 0, 0, 0, 12
50	16	[11]	9, 8, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 8, 9
		[12]	17, 17, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
		[13]	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 17, 17
		[14]	34, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
		[15]	0, 0, 0, 0, 0, 0,0, 0, 0, 0, 0, 0, 0, 0, 0, 34
	8	[16]	11, 10, 0, 0, 0, 0, 10, 11
		[17]	14, 14, 14, 0, 0, 0, 0, 0
		[18]	0, 0, 0, 0, 0, 14, 14, 14
		[19]	42, 0, 0, 0, 0, 0, 0, 0
		[20]	0, 0, 0, 0, 0, 0, 0, 42
100	16	[21]	21, 21, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 21, 21
		[22]	21, 21, 21, 21, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
		[23]	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 21, 21, 21, 21
		[24]	84, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0
		[25]	0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 84
	8	[26]	23, 23, 0, 0, 0, 0, 23, 23
		[27]	23, 23, 23, 23, 0, 0, 0, 0
		[28]	0, 0, 0, 0, 23, 23, 23, 23
		[29]	92, 0, 0, 0, 0, 0, 0, 0
		[30]	0, 0, 0, 0, 0, 0, 0, 92

Table 6. Maximum likelihood and empirical Bayes estimate of scale parameter θ and their estimated risk

C.S. No.	MLE		EWMEL function				MEL function		logarithmic loss function	
			$c = -0.1$		$c = 0.1$		$c = 0$			
	$\hat{\theta}_{ML}$	$ER(\hat{\theta}_{ML})$	$\hat{\theta}_{EBE}$	$ER(\hat{\theta}_{EBE})$	$\hat{\theta}_{EBG}$	$ER(\hat{\theta}_{EBE})$	$\hat{\theta}_{EBG}$	$ER(\hat{\theta}_{EBE})$	$\hat{\theta}_{EBL}$	$ER(\hat{\theta}_{EBL})$
[1]	0.13471	0.00202	0.12930	0.00158	0.13769	0.00228	0.13356	0.00192	0.13410	0.00197
[2]	0.14836	0.00324	0.14259	0.00262	0.15119	0.00357	0.14695	0.00308	0.14763	0.00316
[3]	0.12195	0.00111	0.11689	0.00082	0.12503	0.00130	0.12103	0.00105	0.12145	0.00107
[4]	0.14858	0.00327	0.14280	0.00264	0.15142	0.00359	0.14716	0.00311	0.14785	0.00318
[5]	0.12084	0.00104	0.11581	0.00076	0.12393	0.00102	0.11994	0.00098	0.12035	0.00101
[6]	0.09931	0.00038	0.09434	0.00027	0.10271	0.00047	0.09862	0.00035	0.09893	0.00037
[7]	0.12469	0.00142	0.11852	0.00102	0.12831	0.00167	0.12351	0.00133	0.12409	0.00138
[8]	0.09419	0.00028	0.08947	0.00021	0.09749	0.00035	0.09356	0.00026	0.09385	0.00027
[9]	0.13101	0.00182	0.12454	0.00133	0.12968	0.00171	0.13463	0.00212	0.13034	0.00176
[10]	0.09247	0.00025	0.08783	0.00020	0.09573	0.00031	0.09187	0.00024	0.09214	0.00024
[11]	0.10053	0.00022	0.09613	0.00021	0.10362	0.00029	0.09994	0.00021	0.10020	0.00022
[12]	0.14768	0.00318	0.14192	0.00257	0.15052	0.00350	0.14629	0.00303	0.14696	0.00310
[13]	0.09602	0.00015	0.09178	0.00012	0.09908	0.00019	0.09550	0.00014	0.09572	0.00015
[14]	0.14814	0.00322	0.14237	0.00261	0.15098	0.00355	0.14673	0.00306	0.14741	0.00314
[15]	0.09577	0.00015	0.09153	0.00012	0.09882	0.00019	0.09525	0.00014	0.09547	0.00014
[16]	0.08948	0.00021	0.08500	0.00018	0.09269	0.00026	0.08892	0.00020	0.08917	0.00021
[17]	0.11876	0.00110	0.11286	0.00078	0.12235	0.00131	0.11770	0.00103	0.11821	0.00106
[18]	0.08751	0.00012	0.08312	0.00010	0.09067	0.00012	0.08697	0.00013	0.08721	0.00011
[19]	0.12926	0.00170	0.12287	0.00124	0.12288	0.00158	0.12797	0.00159	0.12861	0.00165
[20]	0.08696	0.00012	0.08260	0.00011	0.09010	0.00011	0.08643	0.00012	0.08667	0.00012
[21]	0.09476	0.00014	0.09057	0.00012	0.09780	0.00017	0.09426	0.00013	0.09447	0.00014
[22]	0.14504	0.00293	0.13936	0.00235	0.14792	0.00324	0.14370	0.00279	0.14435	0.00286
[23]	0.09282	0.00012	0.08870	0.00011	0.09584	0.00015	0.09234	0.00011	0.09254	0.00012
[24]	0.14814	0.00321	0.14137	0.00259	0.15097	0.00355	0.14473	0.00301	0.14541	0.00304
[25]	0.09243	0.00012	0.08832	0.00011	0.09544	0.00014	0.09195	0.00011	0.09215	0.00011
[26]	0.08688	0.00011	0.08252	0.00009	0.09002	0.00012	0.08636	0.00011	0.08659	0.00011
[27]	0.10858	0.00066	0.10317	0.00046	0.11210	0.00080	0.10773	0.00061	0.10813	0.00063

[28]	0.08595	0.00011	0.08163	0.00009	0.08907	0.00011	0.08543	0.00009	0.08566	0.00009
[29]	0.12846	0.00165	0.12210	0.00120	0.12214	0.00132	0.12719	0.00155	0.12781	0.00160
[30]	0.08547	0.00011	0.08118	0.00010	0.08858	0.00010	0.08496	0.00009	0.08519	0.00009

Table 7. Maximum likelihood and empirical Bayes estimates of reliability function $R(t)$ and their estimated risk

C.S. No.	MLE		EWMEL function				MEL function		logarithmic loss function	
			$c=-0.1$		$c=0.1$		$c=0$			
	$\hat{R}(t)_{ML}$	$ER(\hat{R}(t)_{ML})$	$\hat{R}(t)_{EBE}$	$ER(\hat{R}(t)_{EBE})$	$\hat{R}(t)_{EBE}$	$ER(\hat{R}(t)_{EBE})$	$\hat{R}(t)_{EBE}$	$ER(\hat{R}(t)_{EBE})$	$\hat{R}(t)_{EBL}$	$ER(\hat{R}(t)_{EBL})$
[1]	0.95523	0.00215	0.95468	0.00210	0.95469	0.00210	0.95468	0.00210	0.95475	0.00211
[2]	0.96384	0.00291	0.96334	0.00286	0.96335	0.00286	0.96335	0.00286	0.96340	0.00286
[3]	0.94526	0.00143	0.94466	0.00139	0.94467	0.00139	0.94467	0.00139	0.94475	0.00140
[4]	0.96396	0.00292	0.96346	0.00287	0.96347	0.00287	0.96346	0.00287	0.96351	0.00288
[5]	0.94427	0.00137	0.94366	0.00133	0.94368	0.00133	0.94367	0.00133	0.94376	0.00134
[6]	0.88963	0.00070	0.88887	0.00068	0.88889	0.00068	0.88889	0.00068	0.88904	0.00068
[7]	0.94435	0.00163	0.94365	0.00158	0.94366	0.00158	0.94365	0.00158	0.94376	0.00159
[8]	0.86773	0.00060	0.86697	0.00059	0.86700	0.00059	0.86699	0.00059	0.86714	0.00059
[9]	0.95164	0.00196	0.95096	0.00190	0.95097	0.00190	0.95097	0.00190	0.95106	0.00191
[10]	0.85979	0.00058	0.85903	0.00057	0.85905	0.00057	0.85904	0.00057	0.85920	0.00057
[11]	0.91625	0.00046	0.91556	0.00045	0.91558	0.00045	0.91557	0.00045	0.91570	0.00045
[12]	0.96343	0.00287	0.96293	0.00282	0.96293	0.00282	0.96293	0.00282	0.96299	0.00283
[13]	0.90692	0.00040	0.90621	0.00039	0.90624	0.00039	0.90622	0.00039	0.90637	0.00039
[14]	0.96371	0.00291	0.96321	0.00285	0.96322	0.00285	0.96321	0.00285	0.96326	0.00285
[15]	0.90649	0.00039	0.90578	0.00039	0.90581	0.00039	0.90580	0.00039	0.90595	0.00039
[16]	0.84599	0.00041	0.84524	0.00041	0.84527	0.00041	0.84525	0.00041	0.84542	0.00041
[17]	0.93609	0.00135	0.93537	0.00131	0.93539	0.00131	0.93538	0.00131	0.93549	0.00132
[18]	0.83598	0.00035	0.83523	0.00035	0.83526	0.00035	0.83525	0.00035	0.83541	0.00035
[19]	0.95015	0.00187	0.94946	0.00182	0.94947	0.00182	0.94947	0.00182	0.94956	0.00182
[20]	0.83354	0.00035	0.83279	0.00035	0.83281	0.00035	0.83280	0.00035	0.83297	0.00035
[21]	0.90187	0.00038	0.90115	0.00038	0.90118	0.00038	0.90116	0.00038	0.90132	0.00038
[22]	0.96187	0.00273	0.96136	0.00268	0.96137	0.00268	0.96137	0.00268	0.96142	0.00268
[23]	0.89555	0.00038	0.89484	0.00038	0.89486	0.00038	0.89485	0.00038	0.89501	0.00038

[24]	0.96368	0.00290	0.96318	0.00281	0.96319	0.00281	0.96319	0.00281	0.96314	0.00281
[25]	0.89485	0.00038	0.89412	0.00038	0.89415	0.00038	0.89413	0.00038	0.89429	0.00038
[26]	0.83341	0.00031	0.83265	0.00031	0.83268	0.00031	0.83267	0.00031	0.83284	0.00031
[27]	0.91697	0.00096	0.91622	0.00093	0.91624	0.00093	0.91623	0.00093	0.91637	0.00093
[28]	0.82736	0.00031	0.82662	0.00031	0.82665	0.00031	0.82663	0.00031	0.82680	0.00031
[29]	0.94845	0.00183	0.94776	0.00178	0.94776	0.00178	0.94777	0.00178	0.94786	0.00178
[30]	0.82505	0.00031	0.82430	0.00031	0.82433	0.00031	0.82431	0.00031	0.82448	0.00031

Table 8. Maximum likelihood and empirical Bayes estimates of failure rate function $\lambda(t)$ and their estimated risks

C.S. No.	MLE		EWMEL function				MEL function		logarithmic loss Function	
			$c=-0.1$		$c=0.1$		$c=0$			
	$\hat{\lambda}(t)_{ML}$	$ER(\hat{\lambda}(t)_{ML})$	$\hat{\lambda}(t)_{EBE}$	$ER(\hat{\lambda}(t)_{EBE})$	$\hat{\lambda}(t)_{EBE}$	$ER(\hat{\lambda}(t)_{EBE})$	$\hat{\lambda}(t)_{EBE}$	$ER(\hat{\lambda}(t)_{EBE})$	$\hat{\lambda}(t)_{EBL}$	$ER(\hat{\lambda}(t)_{EBL})$
[1]	2.29327	6.16460	2.21555	6.53985	2.21806	6.52639	2.21681	6.53311	2.31320	6.07123
[2]	1.84206	8.29350	1.76946	8.71323	1.77286	8.69294	1.77116	8.70306	1.85999	8.19177
[3]	2.82035	4.13596	2.73764	4.44650	2.73945	4.43838	2.73855	4.44243	2.84227	4.05720
[4]	1.83599	8.32570	1.76348	8.74591	1.76689	8.72550	1.76518	8.73568	1.85389	8.22391
[5]	2.87306	4.25871	2.78991	4.26642	2.79166	4.25871	2.79078	4.26256	2.89516	3.88602
[6]	3.93842	2.06444	3.83836	2.18939	3.83955	2.18587	3.83896	2.18763	3.96539	2.03743
[7]	2.75345	4.68848	2.65591	5.06926	2.65846	5.05659	2.65719	5.06291	2.77805	4.59823
[8]	4.09846	1.80480	3.99952	1.87942	4.00054	1.87694	4.00003	1.87818	4.12538	1.79092
[9]	2.48476	5.61601	2.38861	6.04559	2.39011	6.03763	2.39161	6.02971	2.50852	5.51499
[10]	4.15094	1.74596	4.05246	1.80310	4.05343	1.80091	4.05294	1.80200	4.17781	1.73672
[11]	4.05745	1.37742	3.96673	1.48969	3.96765	1.48764	3.96719	1.48867	4.08273	1.35047
[12]	1.86344	8.18699	1.79059	8.60476	1.79394	8.58482	1.79227	8.59477	1.88147	8.08560
[13]	4.40401	1.19429	4.31178	1.24173	4.31256	1.24048	4.31217	1.24111	4.42996	1.18545
[14]	1.84902	8.25872	1.77633	8.67781	1.77972	8.65764	1.77803	8.66770	1.86697	8.15710
[15]	4.42745	1.19434	4.33512	1.23746	4.33589	1.23626	4.33550	1.23686	4.45344	1.18672
[16]	4.25102	1.28932	4.15331	1.31411	4.15419	1.31240	4.15375	1.31325	4.27782	1.28874
[17]	3.03276	3.90225	2.93404	4.22933	2.93621	4.21934	2.93512	4.22432	3.05812	3.82461
[18]	4.40779	1.17586	4.41073	1.17999	4.41156	1.17856	4.41114	1.17927	4.43449	1.17284

[19]	2.56422	5.36773	2.46760	5.78344	2.47047	5.76853	2.46904	5.77597	2.58824	5.26977
[20]	4.43259	1.18689	4.43565	1.18381	4.43646	1.18247	4.43605	1.18314	4.45928	1.18384
[21]	4.46256	1.15505	4.37033	1.18847	4.37109	1.18739	4.37071	1.18793	4.48855	1.15013
[22]	1.94472	7.78232	1.87089	8.19276	1.87406	8.17420	1.87248	8.18346	1.96312	7.68218
[23]	4.59040	1.15295	4.49789	1.15920	4.49859	1.15840	4.49824	1.15880	4.61657	1.15270
[24]	1.85015	8.25548	1.77746	8.67437	1.78085	8.65417	1.77916	8.66424	1.86811	8.15392
[25]	4.62959	1.17121	4.53691	1.17007	4.53760	1.16932	4.53726	1.16970	4.65583	1.17607
[26]	4.54005	1.09250	4.44308	1.08784	4.44389	1.08651	4.44349	1.08718	4.56676	1.09988
[27]	3.53445	2.80216	3.43428	3.02499	3.43588	3.01886	3.43508	3.02192	3.56090	2.75020
[28]	4.64298	1.07319	4.54656	1.06206	4.54735	1.06083	4.54695	1.06144	4.66957	1.08228
[29]	2.58748	5.25399	2.49082	5.66474	2.49393	5.44853	2.49223	5.65746	2.61156	5.15708
[30]	4.66097	1.08325	4.56468	1.06634	4.56546	1.06517	4.56507	1.06575	4.68754	1.09393

6.1. SIMULATION RESULTS

From the results of the Monte Carlo simulation study presented in Tables 6 - 8, the following points can be drawn:

1. For the fixed sample size n, as the effective sample size m decreases, estimated risks of the maximum likelihood and empirical Bayes estimates of reliability performances decreases, i.e., the performance of the estimates becomes better with decreasing effective sample sizes in terms of estimated risk.
2. For the fixed effective sample size m, as the sample size n increases, estimated risks of the maximum likelihood and empirical Bayes estimates of reliability performances decreases, i.e., the performance of the estimates becomes better with increasing sample sizes in terms of estimated risk.
3. The estimated risks of empirical Bayes estimates are smaller than that of corresponding maximum likelihood estimates in all the considered cases. This means that the performance of the empirical Bayes estimates is better than their corresponding maximum likelihood estimates in terms of estimated risk.
4. The maximum likelihood and empirical Bayes estimates of reliability performances have the smallest estimated risk for the conventional Type II censoring scheme in most of the considered cases, i.e., this censoring scheme is the most efficient one compared to other schemes.

5. The performance of the maximum likelihood and empirical Bayes estimates (under MEL and logarithmic loss functions) of parameter θ are very similar in nature in terms of absolute bias.
6. The empirical Bayes estimates of θ under EWMEL function are sensitive to the value of shape parameter c, whereas of $R(t)$ and $\lambda(t)$ are not.
7. For the fixed sample size n, as the effective sample size m decreases, the absolute biases of maximum likelihood and empirical Bayes estimates of $\lambda(t)$ decreases. Furthermore, for the fixed effective sample size m, as the fixed sample size n increases, the absolute bias of maximum likelihood and empirical Bayes estimates of $\lambda(t)$ decreases.
8. Different values of the hyper-parameter b and mission time t have been examined, and almost the same conclusions stated above are observed.

7. CONCLUSION

Progressive Type II censoring has received considerable attention in life testing and reliability studies since the last few years, due to the availability of high-speed computing resources. Under this censoring scheme, the Bayes and empirical Bayes estimates of the reliability performances of the Rayleigh model are obtained in the closed form with respect to EWMEL and logarithmic loss functions. A real data set has been analyzed for an illustrative purpose. The simulation study is carried out to examine and compare the performance of Maximum likelihood and empirical Bayes estimates in terms of bias and estimated risk for different sample sizes, effective sample sizes, and progressive censoring schemes with five withdrawal patterns.

The simulation results show that the proposed empirical Bayes estimates perform better than their corresponding maximum likelihood estimates for all the considered cases. Moreover, the estimated risks of the estimates get smaller with decreasing ratio m/n.

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