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NEIGHBORHOOD OF A CLASS OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY THE GENERALIZED RUSCHEWEY DERIVATIVES INVOLVING A GENERAL FRACTIONAL DERIVATIVE OPERATOR

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ABSTRACT. By making use of the familiar concept of neighborhoods of analytic functions, we prove several inclusion relations associated with the (n, δ)-neighborhoods of various subclasses of starlike and convex functions of complex order defined by the generalized Ruscheweyh derivative involving a general fractional derivative operator. Special cases of some of these inclusion relations are shown to yield known results.

INTRODUCTION

Let $A(n)$ denote the class of functions $f(z)$ of the form

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$$f(z)=z-\sum_{k=n+1}^{\infty}a_kz^k(a_k\geq 0;n\in\mathbb{N}=\{1,2,3,...\}), \quad (1)$$

which are analytic in the open unit disk

$$\mathcal{U}=\{z\in\mathbb{C}:|z|<1\}.$$

Following [4, 8], we define the (n,δ)-neighborhood of a function $f(z)\in A(n)$ by

$$N_{n,\delta}(f)=\{g\in A(n):g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k and \sum_{k=n+1}^{\infty}k|a_k-b_k|\leq\delta\}. \quad (2)$$

In particular, for the identity function

$$e(z)=z,$$

we immediately have

$$N_{n,\delta}(e)=\{g\in A(n):g(z)=z-\sum_{k=n+1}^{\infty}b_kz^k and \sum_{k=n+1}^{\infty}k|b_k|\leq\delta\}. \quad (3)$$

The main object of the present paper is to investigate the (n,δ)-neighborhoods of the following subclasses of the class $A(n)$ of normalized analytic functions in $\mathcal{U}$ with negative coefficients.

A function $f(z)\in A(n)$ is said to be starlike of complex order $\gamma(\gamma\in\mathbb{C}-\{0\})$, that is, $f\in S_n^*(\gamma)$, if it also satisfies the inequality

$$\Re\{1+\frac{1}{\gamma}(\frac{zf'(z)}{f(z)}-1)\}>0,(z\in\mathcal{U};\gamma\in\mathbb{C}-\{0\}).$$

Furthermore, a function $f(z) \in A(n)$ is said to be convex of complex order $\gamma(\gamma \in \mathbb{C} - \{0\})$, that is, $f \in C_n(\gamma)$, if it satisfies the inequality

$$\Re\{1 + \frac{1}{\gamma}\frac{zf''(z)}{f'(z)})\} > 0, (z \in \mathcal{U}; \gamma \in \mathbb{C} - \{0\}).$$

The classes $S_n^*(\gamma)$ and $C_n(\gamma)$ stem essentially from the classes of starlike and convex functions of complex order, which were considered earlier by Nasr and Aouf [6] and Wiatrowski[13], respectively, (see also [3, 11]).

We shall need the fractional derivative operator ([9], [12]) in this paper.

Let $a, b, c \in \mathbb{C}$ with $\mathbb{C} \neq \{0, -1, -2, ...\}$. The Gaussian hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(z) ={}_2 F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!},$$

where $(\lambda)_n$ is the Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1)...(\lambda + n - 1) & (n \in \mathbb{N}). \end{cases}$$

Definition 1.1: Let $0 \leq \eta < 1$ and $\mu, v \in \mathbb{R}$. Then, in terms of familiar (Gauss's) hypergeometric function ${}_2F_1$, the generalized fractional derivative operator $J_{0,z}^{\eta,\mu,v}$ of a function $f(z)$ is defined by:

$$J_{0,z}^{\eta,\mu,v} f(z) = \begin{cases} \frac{1}{\Gamma(1-\eta)}\frac{d}{dz}\{z^{\eta-\mu}\int_0^z (z-\epsilon)^{-\eta} f(\epsilon).{}_2F_1(\mu - \eta, 1 - v; 1 - \eta; 1 - \frac{\epsilon}{z})d\epsilon\} \\ (0 \leq \eta < 1) \\ \frac{d^n}{dz^n} J_{0,z}^{\eta-n,\mu,v} f(z), (n \leq \eta < n + 1, n \in \mathbb{N}) \end{cases} \quad (4)$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin, with the order

$$f(z) = O(|z|^{\epsilon}), (z \to 0),$$

for $\epsilon > \max\{0, \mu - v\} - 1$, and the multiplicity of $(z-\epsilon)^{-\eta}$ is removed by requiring $\log(z-\epsilon)$ to be real, when $z - \epsilon > 0$.

The fractional derivative of order η of a function $f(z)$ is defined by

$$D_z^{\eta}\{f(z)\} = \frac{1}{\Gamma(1-\eta)} \frac{d}{dz} \int_0^z \frac{f(\epsilon)}{(z-\epsilon)^{\eta}} d\epsilon, 0 \le \eta < 1, \tag{5}$$

where $f(z)$ it is chosen as in (4), and the multiplicity of $(z-\epsilon)^{-\eta}$ is removed by requiring $\log(z-\epsilon)$ to be real, when $z - \epsilon > 0$.

By comparing (4) with (5), we find

$$J_{0,z}^{\eta,\eta,v} f(z) = D_z^{\eta}\{f(z)\}, (0 \le \eta < 1).$$

In terms of gamma function, we have

$$J_{0,z}^{\eta,\mu,v} z^k = \frac{\Gamma(k+1)\Gamma(1-\mu+v+k)}{\Gamma(1-\mu+k)\Gamma(1-\eta+v+k)} z^{k-\mu},$$

$(0 \le \eta < 1, \mu, v \in \mathbb{R}$ and $k > \max\{0, \mu - v\} - 1)$.

Now $J_1^{\eta,\mu} f$ is a generalized Ruscheweyh derivatve defined by Goyal and Goyal [5, p. 442] as

$$J_1^{\eta,\mu} f(z) = \frac{\Gamma(\mu-\eta+v+2)}{\Gamma(v+2)\Gamma(\mu+1)} z J_{0,z}^{\eta,\lambda,v}(z^{\mu-1} f(z)), \tag{6}$$

$$= z - \sum_{k=n+1}^{\infty} a_k C_1^{\eta,\mu}(k) z^k,$$

where

$$C_1^{\eta,\mu}(k) = \frac{\Gamma(k+\mu)\Gamma(v+2+\mu-\eta)\Gamma(k+v+1)}{\Gamma(k)\Gamma(k+v+1+\mu-\eta)\Gamma(v+2)\Gamma(1+\mu)}. \tag{7}$$

For $\mu = \eta = \alpha, v = 1$, the generalized Ruscheweyh derivatives of $f(z)$ of order α [7]:

$$D^{\alpha} f(z) = \frac{z}{\Gamma(\alpha+1)} D^{\alpha}(z^{\alpha-1} f(z)) = z - \sum_{k=n+1}^{\infty} a_k C_k(\alpha) z^k,$$

where

$$C_k(\alpha) = \frac{(\alpha+1)(\alpha+2)...(\alpha+k-1)}{(k-1)!}.$$

Finally, let $\mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$|\frac{1}{\gamma}(\frac{z(J_1^{\eta,\mu} f(z))' + \lambda z^2 (J_1^{\eta,\mu} f(z))''}{\lambda z (J_1^{\eta,\mu} f(z))' + (1-\lambda)(J_1^{\eta,\mu} f(z))} - 1)| < \beta \tag{8}$$

$$(z \in \mathcal{U}; \gamma \in \mathbb{C} - \{0\}; 0 \leq \lambda \leq 1; 0 < \beta \leq 1).$$

Also let $\mathfrak{M}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$ denote the subclass of $A(n)$ consisting of functions $f(z)$ which satisfy the inequality

$$|\frac{1}{\gamma}(f'(z) + \lambda z f''(z) - 1)| < \beta \tag{9}$$

$$(z \in \mathcal{U}; \gamma \in \mathbb{C} - \{0\}; 0 \leq \lambda \leq 1; 0 < \beta \leq 1).$$

Various further subclasses of the classes $\mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$ and $\mathfrak{M}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$ with $\gamma = 1, \mu = \eta = 0, v = 1$ were studied in many earlier works (cf., e.g., [2], [10]); see also the references cited in these earlier works). Clearly, we have

$\mathfrak{H}_n^{0,0,1}(\gamma, 0, 1) \subset S_n^*(\gamma)$ and $\mathfrak{M}_n^{0,0,1}(\gamma, 0, 1) \subset C_n(\gamma)$

$$(n \in \mathbb{N}; \gamma \in \mathbb{C} - \{0\}).$$

2. A SET OF INCLUSION RELATIONS INVOLVING $N_{n,\delta}(e)$

In our investigation of the inclusion relations involving $N_{n,\delta}(e)$, we shall require Theorem 2.1 and 2.2 below.

Theorem 2.1 : Let the function $f \in A(n)$ be defined by (1), then $f(z)$ is in the class $\mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$ if and only if

$$\sum_{k=n+1}^{\infty} (\lambda(k-1)+1)(k+\beta|\gamma|-1)C_1^{\eta,\mu}(k)a_k \leq \beta|\gamma|. \tag{10}$$

where $C_1^{\eta,\mu}(k)$ is defined by (7).

Proof: We first suppose that $f \in \mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$. Then, by condition (8), we get:

$$\Re\{\frac{z(J_1^{\eta,\mu}f(z))' + \lambda z^2(J_1^{\eta,\mu}f(z))''}{\lambda z(J_1^{\eta,\mu}f(z))' + (1-\lambda)(J_1^{\eta,\mu}f(z))} - 1\} > -\beta|\gamma|, (z \in \mathcal{U})$$

or equivalently,

$$\Re\{\frac{-\sum_{k=n+1}^{\infty}[\lambda(k-1)+1)(k-1)a_k z^k}{z - \sum_{k=n+1}^{\infty}(\lambda(k-1)+1)a_k z^k}\} > -\beta|\gamma|.(z \in \mathcal{U}) \tag{11}$$

Now choose values of z on the real axis and let $z \to 1^-$ through real values. Then inequality (11) immediately yields the desired condition (10).

Conversely, by applying hypothesis (10) and letting $|z| = 1$, we find that

$$|\frac{z(J_1^{\eta,\mu}f(z))' + \lambda z^2(J_1^{\eta,\mu}f(z))''}{\lambda z(J_1^{\eta,\mu}f(z))' + (1-\lambda)(J_1^{\eta,\mu}f(z))} - 1| = |\frac{\sum_{k=n+1}^{\infty}[\lambda(k-1)+1)(k-1)a_k z^k}{z - \sum_{k=n+1}^{\infty}[\lambda(k-1)+1)a_k z^k}|$$

$$\leq \frac{\beta|\gamma|(1-\sum_{k=n+1}^{\infty}[\lambda(k-1)+1)a_k}{1-\sum_{k=n+1}^{\infty}[\lambda(k-1)+1)a_k}$$

$$< \beta|\gamma|.$$

Hence, by the maximum modulus theorem, we have

$$f \in \mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta).$$

Hence the proof is complete.

Similarly, we can prove the following.

Theorem 2.2 : Let the function $f \in A(n)$ be defined by (1), then $f(z)$ is in the class $\mathfrak{M}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$ if and only if

$$\sum_{k=n+1}^{\infty} k(\lambda(k-1)+1)C_1^{\eta,\mu}(k)a_k \leq \beta|\gamma|. \tag{12}$$

where $C_1^{\eta,\mu}(k)$ is defined by (7).

Remark 2.1: A special case of Theorem 2.1 when $\mu = \eta = 0, v = 1, \gamma = 1$, and $\beta = 1-\alpha, (0 \leq \alpha < 1)$
was given earlier by Altintas [1, p. 489, Theorem 1].

Our first inclusion relation involving $N_{n,\delta}(e)$ is given by the following.

Theorem 2.3 : Let

$$\delta = \frac{(n+1)\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)}, (|\gamma < 1),$$

then

$$\mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta) \subset N_{n,\delta}(e).$$

Proof: For $f \in \mathfrak{H}_n^{\eta,\mu,v}(\gamma, \lambda, \beta)$, Theorem 2.1 immediately yields

$$(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1) \sum_{k=n+1}^{\infty} a_k \leq \beta|\gamma|,$$

so that

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)}. \tag{13}$$

On the other hand, we also find from (10) and (13) that

$$(\lambda n+1)\sum_{k=n+1}^{\infty} ka_k \leq \beta|\gamma|(1-\beta|\gamma|)(\lambda n+1)C_1^{\eta,\mu}(n+1)\sum_{k=n+1}^{\infty} a_k$$

$$\leq \beta|\gamma|(1-\beta|\gamma|)(\lambda n+1)\frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)}$$

$$\leq \frac{(n+1)\beta|\gamma|}{n+\beta|\gamma|C_1^{\eta,\mu}(n+1)}, (|\gamma<1),$$

that is,

$$\sum_{k=n+1}^{\infty} ka_k \leq \frac{(n+1)\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)} = \delta,$$

which, in view of definition (3), proves Theorem 2.1.

By similarly, applying Theorem 2.2 instead of Theorem 2.1, we can prove the following.

Theorem 2.4: Let

$$\delta = \frac{\beta|\gamma|}{(\lambda n+1)C_1^{\eta,\mu}(n+1)},$$

then

$$\mathfrak{M}_n^{\eta,\mu,v}(\gamma,\lambda,\beta) \subset N_{n,\delta}(e).$$

Remark 2.2: A special case of Theorem 2.3 when

$$\gamma = 1-\alpha, (0 \leq \alpha < 1), \mu = \eta = 0, v = 1, \lambda = 0, \beta = 1$$

was given by Altintas and Owa [9, p. 798, Theorem 2.1].

3. NEIGHBORHOODS FOR THE CLASSES $\mathfrak{H}_n^{\eta,\mu,v^{(\tau)}}(\gamma,\lambda,\beta)$ AND $\mathfrak{M}_n^{\eta,\mu,v^{(\tau)}}(\gamma,\lambda,\beta)$

In this section, we determine the neighborhood for each of the classes $\mathfrak{H}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta)$ and $\mathfrak{M}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta)$, which we define as follows. A function $f \in A(n)$ is said to be in the class $\mathfrak{H}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta)$ if there exists a function $g \in \mathfrak{H}_n^{\eta,\mu,v}(\gamma,\lambda,\beta)$ such that

$$|\frac{f(z)}{g(z)} - 1| < 1-\tau, (z \in \mathcal{U}; 0 \le \tau < 1). \tag{14}$$

Analogously, a function $f \in A(n)$ is said to be in the class $\mathfrak{M}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta)$ if there exists a function $g \in \mathfrak{M}_n^{\eta,\mu,v}(\gamma,\lambda,\beta)$ such that inequality (14) holds true.

Theorem 3.1: If $g \in \mathfrak{H}_n^{\eta,\mu,v}(\gamma,\lambda,\beta)$ and

$$\tau = \frac{\delta}{n+1}.\frac{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)-\beta|\gamma|}, \tag{15}$$

then

$$N_{n,\delta}(g) \subset \mathfrak{H}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta).$$

Proof: Suppose $f \in N_{n,\delta}(g)$. We then find from (2) that

$$\sum_{k=n+1}^{\infty} k|a_k - b_k| \le \delta,$$

which readily implies the coefficient inequality

$$\sum_{k=n+1}^{\infty} |a_k - b_k| \le \frac{\delta}{n+1}, (n \in \mathbb{N}).$$

Next, since $g \in \mathfrak{H}_n^{\eta,\mu,v}(\gamma,\lambda,\beta)$, we have:

$$\sum_{k=n+1}^{\infty} a_k \le \frac{\beta|\gamma|}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)},$$

so that

$$|\frac{f(z)}{g(z)} - 1| < \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty} b_k}$$

$$\leq \frac{\delta}{n+1} \cdot \frac{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)}{(\lambda n+1)(n+\beta|\gamma|)C_1^{\eta,\mu}(n+1)-\beta|\gamma|}$$

$$= \tau,$$

provided that τ is given precisely by (15).

Thus, by definition, $f \in \mathfrak{H}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta)$ for τ given by (15).

Hence the proof is complete.

Theorem 3.2: If $g \in \mathfrak{M}_n^{\eta,\mu,v}(\gamma,\lambda,\beta)$ and

$$\tau = \frac{\delta}{n+1} \cdot \frac{(\lambda n+1)(n+1)C_1^{\eta,\mu}(n+1)}{(\lambda n+1)(n+1)C_1^{\eta,\mu}(n+1)-\beta|\gamma|}$$

then

$$N_{n,\delta}(g) \subset \mathfrak{M}_n^{\eta,\mu,v(\tau)}(\gamma,\lambda,\beta).$$

The proof is similar to that of Theorem 3.1, hence it is omitted.

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