ON THE STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS IN PARTIALLY ORDERED BANACH SPACES: A PARTIALLY ORDERED FIXED POINT APPROACH

MARYAM RAMEZANI (1) AND HAMID BAGHANI (2)

Abstract. Using partially ordered fixed point method, we investigate the Hyers-Ulam-Rassias stability and superstability of quadratic functional equations on Banach spaces endowed a partial order.

1. INTRODUCTION

The problem of the stability of functional equations was originally stated by S. M. Ulam [1]. In 1941 D.H. Hyers [2] proved the stability of the additive functional equation for the special case when the groups $G_1$ and $G_2$ are Banach spaces. In 1950, T. Aoki discussed the Hyers-Ulam stability theorem in [3]. His result was further generalized and derived as a special case by Th.M. Rassias [4] in 1978. The stability problem for functional equations have been extensively investigated by a number of mathematicians [5, 6, 7, 8, 9, 10, 11]. The quadratic function $f(x) = cx^2$ satisfies the functional equation

\begin{equation}
 f(x + y) + f(x - y) = 2f(x) + 2f(y)
\end{equation}

and therefore the equation (1.1) is called the quadratic functional equation. The Hyers-Ulam stability theorem for the quadratic functional equation (1.1) was proved
by F. Skof [10] for the functions $f : E_1 \to E_2$ where $E_1$ is a normed space and $E_2$ is a Banach space. The result of Skof is still true if the relevant domain $E_1$ is replaced by an Abelian group and this was dealt with by P.W. Cholewa [12]. S. Czerwik [13] proved the Hyers-Ulam-Rassias stability of the quadratic functional equation (1.1). This result was further generalized by Th.M. Rassias [14], C. Borelli and G.L. Forti [15].

In this paper, we will adopt the fixed point alternative of [16], to prove the Hyers-Ulam-Rassias stability and superstability of quadratic mappings on Banach spaces endowed a partial order and associated with the following generalized quadratic type functional equation

$$(1.2) f(x_1 + x_2 + x_3 - x_4) + f(x_1 - x_2 - x_3 + x_4) = 2f(x_1) + 2f(x_2 + x_3 - x_4)$$

2. Stability

In this section and next section, we suppose that $(E_1, \|\cdot\|_1)$ is a normed space endowed with a partial order $\leq_1$ with following conditions:

(i) $x, y \in E_1$ and $x \leq_1 y \Rightarrow rx \leq_1 ry$ $\forall r \in \mathbb{R}^+$;

(ii) for all $x, y \in E_1$ there exists $z \in E_1$ such that $z$ is comparable to $x$ and $y$.

Also, we suppose that $(E_2, \|\cdot\|_2)$ is a Banach space endowed with a partial order $\leq_2$ which satisfies condition (i) and

(iii) for all $x, y \in E_1$ there exists $z \in E_1$ such that $z$ is an upper bound for $\{x, y\}$.

(iv) If $\{x_n\}$ is a nondecreasing sequence in $E_2$ and $x_n \to x$, then $x \geq x_n$ for all $n \in \mathbb{N}$.

As a example, we can see that the set

$$C([0, 1]) := \{f : [0, 1] \to \mathbb{R} \mid f \text{ is continuous}\}$$
with the following partial order

\[ f, g \in C([0, 1]), \quad f \leq g \iff f(x) \leq g(x) \text{ for all } 0 \leq x \leq 1. \]

It is easy to show that for any \( f, g \in C([0, 1]) \) the function \( \max\{f, g\} \) is upper bound of \( f \) and \( g \).

In this section, we consider \( 0 \times \infty = 0 \). Before of our main results we need the following proposition.

**Lemma 2.1.** The functional equation (1.2) is a quadratic functional equation.

**Proof.** By letting \( x_3 = x_4 := 0 \) in (1.2) we get

\[ f(x_1 + x_2) + f(x_1 - x_2) = 2f(x_1) + 2f(x_2) \]

and this shows that (1.2) is a quadratic functional equation. Also, putting \( x := x_1 \) and \( y := x_2 + x_3 - x_4 \) in (1.1) we infer the equation (1.2). \( \square \)

**Theorem 2.1.** Suppose \( f : E_1 \to E_2 \) is a function satisfies

(2.1) \hspace{1cm} 4f(x) \leq_2 f(2x) \quad (x \in E_1)

and

\[ \|f(x + y + z - w) + f(x - y - z + w) - 2f(x) - 2f(y + z - w)\|_2 \leq \phi(x, z) + \phi(y, w) \]

(2.2) \hspace{1cm} \leq \phi(x, z) + \phi(y, w)

for all \( x, y, z, w \in E_1 \) which \( x \) is comparable to \( z \) and \( y \) is comparable to \( w \), where

\( \phi : E_1 \times E_1 \to [0, \infty) \) is a function satisfies \( \phi(0, 0) = 0 \) and with the following condition:

(2.3) \hspace{1cm} \phi(x, y) \leq 4 L \phi(x, y) \phi(x, y) \frac{x}{2} \cdot \frac{y}{2} \]

for all \( x, y, z, w \in E_1 \) which \( x \) is comparable to \( z \) and \( y \) is comparable to \( w \), where
for all $x, y \in E_1$ with $x$ comparable to $y$, which $L \in (0, 1)$ is a constant. Then there exists a unique a quadratic mapping $H : E_1 \to E_2$ such that

\begin{equation}
\|H(x) - f(x)\|_2 \leq \frac{1}{1 - L} \phi(x, x)
\end{equation}

for all $x \in E_1$.

**Proof.** We can see that $f(0) = 0$. Putting $z := x$ and $y = w := 0$ in (2.9), we get

\[\|f(2x) - 4f(x)\|_2 \leq \phi(x, x)\]

for all $x \in E_1$. Hence

\begin{equation}
\left\|\frac{f(2x)}{4} - f(x)\right\|_2 \leq \frac{1}{4} \phi(x, x) \leq \phi(x, x)
\end{equation}

for all $x \in E_1$. We consider the set $X := \{g \mid g : E_1 \to E_2\}$ and introduce the metric $d$ on $X$ by:

\[d(h, g) := \inf \{C \in \mathbb{R}^+ ; \|h(x) - g(x)\|_2 \leq C \phi(x, x) \text{ for all } x \in E_1\}\]

for all $h, g \in X$. It is easy to show that $(X, d)$ is a complete generalized metric space. We put the partial order $\leq$ on $X$ as follows:

\[h, g \in X \quad h \leq g \iff h(x) \leq g(x) \text{ for all } x \in E_1.\]

Now, we define the mapping $J : X \to X$ by

\[J(h)(x) := \frac{1}{4} h(2x)\]
for all $x \in E_1$. For any $g, h \in X$ with $g \leq h$, we have

$$d(g, h) < C \Rightarrow \|g(x) - h(x)\|_2 \leq C \phi(x, x) \text{ for all } x \in E_1$$

$$\Rightarrow \left\| \frac{g(2x)}{4} - \frac{h(2x)}{4} \right\|_2 \leq C \frac{\phi(2x, 2x)}{4} \text{ for all } x \in E_1$$

$$\Rightarrow \|J(g)(x) - J(h)(x)\|_2 \leq L C \phi(x, x) \text{ for all } x \in E_1.$$ It follows that

$$d(J(g), J(h)) \leq L d(g, h).$$

Applying inequalities (2.8) and (2.12), we can see that $f \leq J(f)$ and $d(J(f), f) \leq 1$, also, using the condition $(i)$ of $E_1$ we can show that $J$ is a nondecreasing mapping. Now, we show that $J$ is a continuous function. To this end, let $\{h_n\}$ be a sequence in $(X, d)$ such that converges to $h \in X$ and let $\epsilon > 0$ be given. Then there exist $N \in \mathbb{N}$ and $C \in \mathbb{R}^+$ with $C \leq \epsilon$ such that

$$\|h_n(x) - h(x)\|_2 \leq C \phi(x, x)$$

for all $x \in E_1$ and all $n \geq N$. Thus we get

$$\|h_n(2x) - h(2x)\|_2 \leq C \phi(2x, 2x)$$

for all $x \in E_1$ and all $n \geq N$. By inequality (2.10) and definition of $J$, we get

$$\|J(h_n)(x) - J(h)(x)\|_2 \leq L C \phi(x, x)$$

for all $x \in E_1$ and $n \geq N$. Hence,

$$d(J(h_n), J(h)) \leq L C < \epsilon$$
for all $n \geq N$. It follows that $J$ is continuous. Applying Theorem ??, we get $J$ has a fixed point. Let $T \in X$ is a fixed point of $J$, then $\lim_{n \to \infty} d(J^n(f), T) = 0$. It follows that

\begin{equation}
(2.6) \quad H(x) = \lim_{n \to \infty} \frac{f(2^n x)}{4^n}
\end{equation}

for all $x \in E_1$. On the other hand, it follows from (2.8) that for all $x \in E_1$, the sequence $\left\{ \frac{f(2^n x)}{4^n} \right\}_{n=0}^{\infty}$ is a nondecreasing sequence in $E_2$, hence, by using the condition $(iii)$, we find that $f(x) \leq T(x)$, for all $x \in E_1$. This shows that $f \leq T$. Now, we can see that

\[ d(J(f), J(T)) \leq L \, d(f, T) \]

and hence

\[ d(f, T) \leq \frac{1}{1 - L}. \]

This implies the inequality (2.11). The inequality (2.10) shows that

\begin{equation}
(2.7) \quad 4^{-n} \, \phi(2^n x, 2^n y) \leq L^n \, \phi(x, y)
\end{equation}

for all $x, y \in E_1$ which $x$ is comparable to $y$ and for all $n \in \mathbb{N}$. Let $x, y \in E_1$ are arbitrary elements, then there exists $z \in E_1$ such that $z$ is comparable to $x$ and $y$. This implies that $2^n z$ is comparable to $2^n x$ and $2^n y$ for all $n \in \mathbb{N}$. It follows from (2.9) that

\[ \|f(2^n(x+y)) + f(2^n(x-y)) - 2f(2^n x) - 2f(2^n y)\|_2 \]

\[ = \|f(2^n x + 2^n y + 2^n z - 2^n z) + f(2^n x - 2^n y - 2^n z + 2^n z) \]

\[ - f(2^n x) - f(2^n y + 2^n z - 2^n z)\|_2 \]

\[ \leq \phi(2^n x, 2^n z) + \phi(2^n y, 2^n z) \]

for all $n \in \mathbb{N}$. Since $L \in (0, 1)$ and by using (2.13) and (2.14), we find that $H$ is a Cauchy mapping. To prove the uniqueness property of $H$, we suppose that $T_1$ is another quadratic function satisfying (2.11). It is clear that $J(T_1) = T_1$. 
Fix the arbitrary element \( x \in E_1 \), then there exists \( h(x) \in E_2 \) such that \( h(x) = \text{upper bound} \{H(x), T_1(x)\} \). This shows that \( h : E_1 \to E_2 \) is a function comparable to \( H \) and \( T_1 \). Hence,

\[
d(H, T_1) \leq d(H, J^n(h)) + d(J^n(h), T_1) \\
= d(J^n(H), J^n(h)) + d(J^n(h), J^n(T_1)) \\
\leq L^{-n} d(H, h) + L^{-n} d(h, T_1)
\]

for all \( n \in \mathbb{N} \). Therefore, \( T = T_1 \) and this completes the proof. \( \square \)

**Theorem 2.2.** Suppose \( f : E_1 \to E_2 \) is a function satisfies

\[
f(x) \leq 4 f\left(\frac{x}{2}\right) \quad (x \in E_1)
\]

and

\[
\|f(x + y + z - w) + f(x - y - z + w) - 2f(x) - 2f(y + z - w)\|_2 \\
\leq \phi(x, z) + \phi(y, w)
\]

for all \( x, y, z, w \in E_1 \) which \( x \) is comparable to \( z \) and \( y \) is comparable to \( w \), where \( \phi : E_1 \times E_1 \to [0, \infty) \) is a function satisfies \( \phi(0, 0) = 0 \) and with the following condition:

\[
4 L \phi(x, y) \leq \phi(2x, 2y)
\]

for all \( x, y \in E_1 \) with \( x \) comparable to \( y \), which \( L \in (1, \infty) \) is a constant. Then there exists a unique a quadratic mapping \( H : E_1 \to E_2 \) such that

\[
\|H(x) - f(x)\|_2 \leq \frac{1}{L - 1} \phi(x, x)
\]

for all \( x \in E_1 \).
Proof. It is clearly that $f(0) = 0$. Putting $z = x := \frac{x}{2}$ and $y = w := 0$ in (2.9), we get
\[
\|f(x) - 4 f(\frac{x}{2})\| \leq \phi(\frac{x}{2}, \frac{x}{2})
\]
for all $x \in E_1$. Hence
\[
\|f(x) - 4 f(\frac{x}{2})\| \leq \frac{1}{4L} \phi(x, x) \leq \frac{1}{L} \phi(x, x)
\]
for all $x \in E_1$. We consider the set $X := \{g| g : E_1 \to E_2\}$ and introduce the metric $d$ on $X$ by:
\[
d(h, g) := \inf\{C \in \mathbb{R}^+; \|h(x) - g(x)\| \leq C \phi(x, x) \text{ for all } x \in E_1\}
\]
for all $h, g \in X$. It is easy to show that $(X, d)$ is a complete generalized metric space.
We put the partial order $\leq$ on $X$ as follows:
\[
h, g \in X \quad h \leq g \iff h(x) \leq g(x) \text{ for all } x \in E_1.
\]
Now, we define the mapping $J : X \to X$ by
\[
J(h)(x) := 4 h(\frac{x}{2})
\]
for all $x \in E_1$. For any $g, h \in X$ with $g \leq h$, we have
\[
d(g, h) < C \Rightarrow \|g(x) - h(x)\| \leq C \phi(x, x) \text{ for all } x \in E_1
\]
\[
\Rightarrow \|4g(\frac{x}{2}) - 4h(\frac{x}{2})\| \leq 4C \phi(\frac{x}{2}, \frac{x}{2}) \text{ for all } x \in E_1
\]
\[
\Rightarrow \|J(g)(x) - J(h)(x)\| \leq L^{-1} C \phi(x, x) \text{ for all } x \in E_1.
\]
It follows that
\[
d(J(g), J(h)) \leq L^{-1} d(g, h).
\]
Applying inequalities (2.8) and (2.12), we can see that \( f \leq J(f) \) and \( d(J(f), f) \leq L^{-1} \), also, using the condition (i) of \( E_1 \) we can show that \( J \) is a nondecreasing mapping. By the same method of Theorem 2.1 we can prove that \( J \) is a continuous mapping and we get \( J \) has a fixed point. Let \( H \in X \) is a fixed point of \( J \), then \( \lim_{n \to \infty} d(J^n(f), H) = 0 \). It follows that

\[
(2.13) \quad H(x) = \lim_{n \to \infty} 4^n f\left(\frac{x}{2^n}\right)
\]

for all \( x \in E_1 \). On the other hand, it follows from (2.8) that for all \( x \in E_1 \), the sequence \( \{4^n f(\frac{x}{2^n})\}_{n=0}^{\infty} \) is a nondecreasing sequence in \( E_2 \), hence, by using the condition (iii), we find that \( f(x) \leq T(x) \), for all \( x \in E_1 \). This shows that \( f \leq T \). Now, we can see that

\[
d(J(f), J(T)) \leq L^{-1} d(f, T)
\]

and hence

\[
d(f, T) \leq \frac{1}{L - 1}.
\]

This implies the inequality (2.11). The inequality (2.10) shows that

\[
(2.14) \quad \phi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq 4^{-n} L^{-n} \phi(x, y)
\]

for all \( x, y \in E_1 \) which \( x \) is comparable to \( y \) and for all \( n \in \mathbb{N} \). Let \( x, y \in E_1 \) are arbitrary elements, then there exists \( z \in E_1 \) such that \( z \) is comparable to \( x \) and \( y \). This implies that \( \frac{z}{2^n} \) is comparable to \( \frac{x}{2^n} \) and \( \frac{y}{2^n} \) for all \( n \in \mathbb{N} \). It follows from (2.9) that

\[
\|f\left(\frac{x + y}{2^n}\right) + f\left(\frac{x - y}{2^n}\right) - 2f\left(\frac{x}{2^n}\right) - 2f\left(\frac{y}{2^n}\right)\|_2
\]

\[
= \|f\left(\frac{x}{2^n} + \frac{y}{2^n} + \frac{z}{2^n} - \frac{z}{2^n}\right) + f\left(\frac{x}{2^n} - \frac{y}{2^n} - \frac{z}{2^n} + \frac{z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n} + \frac{z}{2^n} - \frac{z}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\|_2
\]

\[
\leq \phi\left(\frac{x}{2^n}, \frac{z}{2^n}\right) + \phi\left(\frac{y}{2^n}, \frac{z}{2^n}\right)
\]
for all $n \in \mathbb{N}$. Since $L^{-1} \in (0,1)$ and by using (2.13) and (2.14), we find that $H$ is a Cauchy mapping. To prove the uniqueness property of $H$, we suppose that $T_1$ is another quadratic function satisfying (2.11). It is clear that $J(T_1) = T_1$. Fix the arbitrary element $x \in E_1$, then there exists $h(x) \in E_2$ such that $h(x) = \text{upper bound} \{H(x), T_1(x)\}$. This shows that $h : E_1 \to E_2$ is a function comparable to $H$ and $T_1$. Hence,

\[
d(H, T_1) \leq d(H, J^n(h)) + d(J^n(h), T_1)
= d(J^n(H), J^n(h)) + d(J^n(h), J^n(T_1))
\leq L^{-n} d(H, h) + L^{-n} d(h, T_1)
\]

for all $n \in \mathbb{N}$. Therefore, $H = T_1$ and this completes the proof. \qed

**Corollary 2.1.** Let $\varepsilon \in (0, \infty)$ and $f : E_1 \to E_2$ be a function with $f(0) = 0$ and satisfies the following

\[
4f(x) \leq f(2x) \quad ; \quad (x \in E_1)
\]

\[
\|f(x + y + z - w) + f(x - y - z + w) - 2f(x) - 2f(y + z - w)\| \leq \varepsilon
\]

for all $x, y, z, w \in E_1$ which $x$ is comparable to $z$ and $y$ is comparable to $w$. Then there exists a unique quadratic mapping $H : E_1 \to E_2$ such that

\[
\|H(x) - f(x)\| \leq \varepsilon
\]

for all $x \in E_1$.

Proof. Set $\phi(x, y) = \frac{\varepsilon}{2}$ for all $x, y \in E_1$ with $x, y \neq 0$ and $\phi(0, 0) = 0$ and let $L = \frac{1}{4}$ in Theorem 2.1. Then we get the desired result. \qed
Corollary 2.2. Let $p \in (0, 2)$ ($p \in (2, \infty)$) and $\epsilon \in (0, \infty)$ are real numbers. Suppose that $f : E_1 \rightarrow E_2$ is a mapping satisfies

$$4f(x) \leq_2 f(2x) \quad (f(x) \leq_2 4f(\frac{x}{2})) \quad ; \quad (x \in E_1)$$

for all $x, y, z, w \in E_1$ which $x$ is comparable to $z$ and $y$ is comparable to $w$. Then there exists a unique quadratic mapping $H : E_1 \rightarrow E_2$ such that

$$\|H(x) - f(x)\|_2 \leq \frac{2^{3-p}}{2^{2-p} - 1} \epsilon \|x\|^p \quad (\|H(x) - f(x)\|_2 \leq \frac{2}{2^{p-2} - 1} \epsilon \|x\|^p)$$

for all $x \in E_1$.

Proof. Set $\phi(x, y) = \epsilon (\|x\|^p + \|y\|^p)$ for all $x, y \in E_1$ and $L = 2^{p-2}$ ($L = 2^{p-2}$) in Theorem 2.1 (2.2). Then we get the desired result. \qed

References


(1) Department of Mathematics, University of Bojnord, Bojnord, Iran
E-mail address: m.ramezani@ub.ac.ir

(2) Department of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran.
E-mail address: h.Baghani@gmail.com