UNIQUENESS OF ENTIRE FUNCTIONS CONCERNING PRODUCT OF DIFFERENCE POLYNOMIALS

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ABSTRACT. In this paper, using the concept of weakly weighted sharing and relaxed weighted sharing we investigate the uniqueness of product of difference polynomials that share a small function. The results of the paper improve and extend the recent results due to Chao Meng [9].

1. INTRODUCTION AND DEFINITIONS

A meromorphic function $f$ means meromorphic in the complex plane. If no poles occur, then $f$ is called an entire function. The fundamental results and the standard basics of the Nevanlinna value distribution theory of entire functions are used (see [4],[11],[14]). For a meromorphic function $f$, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all $r$ outside a possible exceptional set of the finite logarithmic measure.

Let $a$ be a finite complex number, and $l$ be a positive integer. We denote by $N_l(r, \frac{1}{f-a})$ the counting function for the zeros of $f(z) - a$ with multiplicity $\leq l$, and by $\overline{N}_l(r, \frac{1}{f-a})$ the corresponding one for which multiplicity is not counted.

Let $N_l(r, \frac{1}{f-a})$ be the counting function for the zeros of $f(z) - a$ with multiplicity $\geq l$ and $\overline{N}_l(r, \frac{1}{f-a})$ be the corresponding one for which multiplicity is not counted. Moreover, we set $N_l(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_2(r, \frac{1}{f-a}) + ... + \overline{N}_l(r, \frac{1}{f-a})$. In the same

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way, we can define $N_l(r, f)$.

Recently, A. Banerjee and S. Mukherjee \[1\] introduced another sharing notion which is also a scaling between IM and CM but weaker than weakly weighted sharing.

**Definition 1.** \[1\] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N_E(r, a; f, g)(N_E(r, a; f, g))$ the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ with same multiplicities and by $N_0(r, a; f, g)(N_0(r, a; f, g))$ the counting function (reduced counting function) of all common zeros of $f - a$ and $g - a$ ignoring multiplicities. If

$$\overline{N}(r, \frac{1}{f - a}) + \overline{N}(r, \frac{1}{g - a}) - 2N_E(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that $f$ and $g$ share the value $a$ “CM”. If

$$\overline{N}(r, \frac{1}{f - a}) + \overline{N}(r, \frac{1}{g - a}) - 2N_0(r, a; f, g) = S(r, f) + S(r, g),$$

then we say that $f$ and $g$ share the value $a$ “IM”.

**Definition 2.** \[7\] Let $f$ and $g$ share the value $a$ “IM” and $k$ be a positive integer or infinity. Then $N_E^k(r, a; f, g)$ denotes the reduced counting function of those $a$-points of $f$ whose multiplicities are equal to the corresponding $a$-points of $g$, and both of their multiplicities are not greater than $k$. $N_0^k(r, a; f, g)$ denotes the reduced counting function of those $a$-points of $f$ which are $a$-points of $g$ and both of their multiplicities are not less than $k$.

**Definition 3.** \[7\] For $a \in \mathbb{C} \cup \{\infty\}$, if $k$ is a positive integer or $\infty$ and

$$\overline{N}_k(r, \frac{1}{f - a}) - N_E^k(r, a; f, g) = S(r, f),$$

$$\overline{N}_k(r, \frac{1}{g - a}) - N_E^k(r, a; f, g) = S(r, g),$$

$$\overline{N}_{(k+1)}(r, \frac{1}{f - a}) - N_0^{(k+1)}(r, a; f, g) = S(r, f),$$

$$\overline{N}_{(k+1)}(r, \frac{1}{g - a}) - N_0^{(k+1)}(r, a; f, g) = S(r, g),$$
or if \( k = 0 \) and
\[
\mathcal{N} \left( r, \frac{1}{f - a} \right) - \mathcal{N}_0(r, a; f, g) = S(r, f),
\]
\[
\mathcal{N} \left( r, \frac{1}{g - a} \right) - \mathcal{N}_0(r, a; f, g) = S(r, g),
\]
then we say \( f \) and \( g \) weakly share \( a \) with weight \( k \). Here we write \( f, g \) share \( (a, k) \) to mean that \( f, g \) weakly share \( a \) with weight \( k \).

**Definition 4.** We denote by \( \mathcal{N}(r, a; f \mid = p; g \mid = q) \) the reduced counting function of common \( a \)-points of \( f \) and \( g \) with multiplicities \( p \) and \( q \), respectively.

**Definition 5.** Let \( f, g \) share a “IM.” Also let \( k \) be a positive integer or \( \infty \) and \( a \in \mathbb{C} \cup \{\infty\} \). If \( \sum_{p,q \leq k} \mathcal{N}(r, a; f \mid = p; g \mid = q) = S(r) \), then we say \( f \) and \( g \) share \( a \) with weight \( k \) in a relaxed manner. Here we write \( f \) and \( g \) share \( (a, k)^* \) to mean that \( f \) and \( g \) share \( a \) with weight \( k \) in a relaxed manner.

In 1997, Yang and Hua [12], studied the unicity of differential monomials and obtained the following theorem.

**Theorem 1.1.** Let \( f(z) \) and \( g(z) \) be two non-constant entire functions, \( n \geq 6 \) a positive integer. If \( f^n f' \) and \( g^n g' \) share 1 CM, then either \( f(z) = c_1 e^{cz} \), \( g(z) = c_2 e^{-cz} \), where \( c_1, c_2, c \) are three constants satisfying \((c_1 c_2)^{n+1} c^2 = -1\) or \( f(z) \equiv tg(z) \) for a constant \( t \) such that \( t^{n+1} = 1 \).

In 2001, Fang and Hong studied the unicity of differential polynomials of the form \( f^n(f - 1)f' \) and proved the following uniqueness theorem.

**Theorem 1.2.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions, \( n \geq 11 \) an integer. If \( f^n(f - 1)f' \) and \( g^n(g - 1)g' \) share the value 1 CM, then \( f \equiv g \).
In 2004, Lin and Yi extended the above theorem as to the fixed point. They proved the following result.

**Theorem 1.3.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions, \( n \geq 7 \) an integer. If \( f^n(f - 1)f' \) and \( g^n(g - 1)g' \) share \( z \) CM, then \( f \equiv g \).

**Theorem 1.4.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 7 \) is an integer. If \( f^n(z)(f(z) - 1)f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( \alpha(z) \) CM, then \( f(z) \equiv g(z) \).

In 2014, Chao Meng proved the following results.

**Theorem 1.5.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 7 \) is an integer. If \( f^n(z)(f(z) - 1)f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( "(\alpha(z), 2)" \), then \( f(z) \equiv g(z) \).

**Theorem 1.6.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is non-zero complex constant and \( n \geq 10 \) is an integer. If \( f^n(z)(f(z) - 1)f(z + c) \) and \( g^n(z)(g(z) - 1)g(z + c) \) share \( (\alpha(z), 2) \) then \( f(z) \equiv g(z) \).

**Theorem 1.7.** Let \( f(z) \) and \( g(z) \) be two transcendental entire functions of finite order, and \( \alpha(z) \) be a small function with respect to both \( f(z) \) and \( g(z) \). Suppose that \( c \) is a non-zero complex constant and \( n \geq 16 \) is an integer. If \( \overline{E_2}(\alpha(z), f^n(z)(f(z) - 1)f(z + c)) = \overline{E_2}(\alpha(z), g^n(z)(g(z) - 1)g(z + c)) \) then \( f(z) \equiv g(z) \).

**Question 1.** What can be said about the relationship between two entire functions \( f \) and \( g \) if we consider the difference polynomials of the form \( f^n(z)(f(z) - 1)f'(z) \) and \( g^n(z)(g(z) - 1)g'(z) \)?
1^m \prod_{j=1}^{d} f(z + c_j)^{s_j} \text{ where } n(\geq 1), m(\geq 1) \text{ and } d \geq 1 \text{ are integers?}

In this paper, our main aim is to find the possible answer to above question. We assume, $c_j \in \mathbb{C} \backslash \{0\} (j = 1, 2, ..., d)$ are distinct constants, $n, m, s_j (j = 1, 2, ..., d)$ are positive integers and $\sigma = \sum_{j=1}^{d} s_j = s_1 + s_2 + ... + s_d$.

We prove the following results which improve and extend Theorem 1.5-1.7. The following theorems are the main results of the paper.

2. MAIN RESULTS

**Theorem 2.1.** Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let $c_j (j = 1, 2, ..., d)$ be complex constants and $s_j (j = 1, 2, ..., d)$ be non-negative integers. Suppose $n(\geq 1)$ and $m(\geq 1)$ are integers satisfying $n \geq \sigma + m + 5$. If $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}$ and $g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j}$ share "$(\alpha(z), 2)$", then $f(z) \equiv g(z)$.

**Theorem 2.2.** Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let $c_j (j = 1, 2, ..., d)$ be complex constants and $s_j (j = 1, 2, ..., d)$ be non-negative integers. Suppose $n(\geq 1)$ and $m(\geq 1)$ are integers satisfying $n \geq 2\sigma + 2m + 6$. If $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}$ and $g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j}$ share $(\alpha(z), 2)^*$, then $f(z) \equiv g(z)$.

**Theorem 2.3.** Let $f(z)$ and $g(z)$ be two transcendental entire functions of finite order and $\alpha(z)$ be a small function with respect to both $f(z)$ and $g(z)$. Let $c_j (j = 1, 2, ..., d)$ be complex constants and $s_j (j = 1, 2, ..., d)$ be non-negative integers. Suppose $n(\geq 1)$ and $m(\geq 1)$ are integers satisfying $n \geq 4\sigma + 4m + 8$. If $\overline{E}_2(\alpha(z), f^n(f(z)-1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}) = \overline{E}_2(\alpha(z), g^n(g(z)-1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j})$, then $f(z) \equiv g(z)$. 


Remark 2.1. Since Theorems 1.5-1.7 can be obtained from Theorems 2.1-2.3 respectively by putting \( m = 1 \) and \( \sigma = 1 \), Theorems 2.1-2.3 improve and extend Theorems 1.5-1.7 respectively.

3. Lemmas

Let \( F \) and \( G \) be two non-constant meromorphic functions defined in \( \mathbb{C} \). We denote by \( H \) the function as follows.

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right)
\]

Lemma 3.1. Let \( H \) be defined as above. If \( F \) and \( G \) share \( "(1, 2)" \) and \( H \neq 0 \), then

\[
T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) - \sum_{p=3}^{\infty} N_p(r, \frac{G}{F}) + S(r, F) + S(r, G),
\]

and the same inequality holds for \( T(r, G) \).

Lemma 3.2. Let \( H \) be defined as above. If \( F \) and \( G \) share \( (1, 2)^* \) and \( H \neq 0 \), then

\[
T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + N_2(r, F) + N_2(r, G) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) - m(r, \frac{1}{G - 1})
\]

\[
+ S(r, F) + S(r, G),
\]

and the same inequality holds for \( T(r, G) \).

Lemma 3.3. Let \( H \) be defined as above. If \( H \equiv 0 \) and

\[
\limsup_{r \to \infty} \frac{\overline{N}(r, \frac{1}{F}) + \overline{N}(r, F) + \overline{N}(r, \frac{1}{G}) + \overline{N}(r, G)}{T(r)} < 1, \ r \in I
\]

where \( T(r) = \max\{T(r, F), T(r, G)\} \) and \( I \) is a set with infinite linear measure, then \( F \equiv G \) or \( FG \equiv 1 \).

Lemma 3.4. Let \( F \) and \( G \) be two non-constant entire functions, and \( p \geq 2 \) an integer. If \( \overline{E}_p(1, F) = \overline{E}_p(1, G) \) and \( H \neq 0 \), then

\[
T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + 2\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) + S(r, F) + S(r, G).
\]
Lemma 3.5. Let \( f(z) \) be a meromorphic function in the complex plane of finite order \( \rho(f) \), and let \( \eta \) be a fixed non-zero complex number. Then for each \( \epsilon > 0 \) one has
\[
T(r, f(z + \eta)) = T(r, f(z)) + O(r^{\rho(f) - 1 + \epsilon}) + O(\log r).
\]

Lemma 3.6. Let \( f(z) \) be a entire function of finite order \( \rho(f) \), c a fixed non-zero complex number, and \( P(z) = a_n f^n(z) + a_{n-1} f^{n-1}(z) + \ldots + a_1 f(z) + a_0 \) where \( a_j (j = 0, 1, \ldots, n) \) are constants. If \( F(z) = P(z) f(z + c) \), then
\[
T(r, F) = (n + 1) T(r, f) + O(r^{\rho(f) - 1 + \epsilon}) + O(\log r).
\]

Lemma 3.7. Let \( f \) be meromorphic function of finite order and \( c \) be a non-zero complex constant. Then,
\[
m \left( r, \frac{f(z + c)}{f(z)} \right) + m \left( r, \frac{f(z)}{f(z + c)} \right) = O\{r^{\rho(f) - 1 + \epsilon}\}.
\]

Lemma 3.8. Let \( f \) be an entire function of order \( \rho(f) \) and \( F(z) = f^n(z)(f(z) - 1)^m \prod_{j=1}^d f(z + c_j)^{s_j} \) where \( n (\geq 1) \) and \( m (\geq 1) \) are integers. Then,
\[
T(r, F) = (n + m + \sigma) T(r, f) + O\{r^{\rho(f) - 1 + \epsilon}\} + S(r, f),
\]
for all \( r \) outside of a set of finite linear measure where \( \sigma = s_1 + s_2 + \ldots + s_d = \sum_{j=1}^d s_j \).

Proof. Since \( f \) is an entire function of finite order, from Lemma 3.7 and standard Valiron-Mohon’ko theorem we have
\[
(n + m + \sigma) T(r, f(z)) = T(r, f^{n+\sigma}(z)(f(z) - 1)^m) + S(r, f)
\]
\[
= m \left( r, \frac{f^{n+\sigma}(z)(f(z) - 1)^m}{f(z)} \right) + S(r, f)
\]
\[
\leq m \left( r, \frac{f^{n+\sigma}(z)(f(z) - 1)^m}{F(z)} \right) + m(r, F(z)) + S(r, f)
\]
\[
\leq m \left( r, \frac{f^{\sigma}(z)}{\prod_{j=1}^d f(z + c_j)^{s_j}} \right) + m(r, F(z)) + S(r, f)
\]
\[
\leq T(r, F(z)) + O\{r^{\rho(f) - 1 + \epsilon}\} + S(r, f). \tag{3.1}
\]
On the other hand, from Lemma 3.5, we have

\[ T(r, F(z)) \leq m(r, f^n(z)) + m(r, (f(z) - 1)^m) + m \left( r, f^\sigma(z) \cdot \prod_{j=1}^{d} \frac{f(z + c_j)^{s_j}}{(f(z))^{s_j}} \right) + S(r, f) \]

\[ \leq (n + m) m(r, f(z)) + \sigma m(r, f(z)) + \sum_{j=1}^{d} s_j \cdot m \left( r, \frac{f(z + c_j)}{f(z)} \right) + S(r, f) \]

\[ \leq (n + m + \sigma) m(r, f(z)) + O \{ r^{\rho(f) - 1 + \varepsilon} \} + S(r, f) \]

(3.2)

From 3.1 and 3.2, we can prove this lemma easily.

4. PROOF OF THE THEOREMS

Proof of Theorem 2.1.

Let \( F(z) = \frac{(f(z))^{(f(z)-1)^m} \prod_{j=1}^{d} f(z+c_j)^{s_j}}{\alpha(z)} \), \( G(z) = \frac{(g(z))^{(g(z)-1)^m} \prod_{j=1}^{d} g(z+c_j)^{s_j}}{\alpha(z)} \).

Then \( F(z) \) and \( G(z) \) share “(1, 2)” except the zeros or poles of \( \alpha(z) \). By Lemma 3.6, we have

\[ T(r, F(z)) = T(r, f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}) + S(r, f) \]

(4.1)

\[ T(r, G(z)) = T(r, g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j}) + S(r, g) \]

(4.2)

Also, we have

\[ N_2(r, \frac{1}{F}) = N_2(r, \frac{1}{f^n(f - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}}) + S(r, f) \]

\[ = N_2(r, \frac{1}{f^n}) + N(r, \frac{1}{(f - 1)^m}) + N(r, \frac{1}{\prod_{j=1}^{d} f(z + c_j)^{s_j}}) + S(r, f) \]

\[ \leq (2 + m + \sigma) T(r, f) + S(r, f) \]

(4.3)

and

\[ N_2(r, \frac{1}{G}) \leq (2 + m + \sigma) T(r, g) + S(r, g) \]

(4.4)
Suppose $H \not\equiv 0$, then by Lemmas 3.1, 3.5 and Lemma 3.8, we have

\[
T(r, F) + T(r, G) \leq 2N_2(r, \frac{1}{F}) + 2N_2(r, \frac{1}{G}) + S(r, f) + S(r, g)
\]

\[
\leq 4\overline{N}(r, \frac{1}{f}) + 2N(r, \frac{1}{(f-1)m}) + 2N(r, \frac{1}{\prod_{j=1}^d f(z + c_j)^{s_j}})
\]

\[
+ 4\overline{N}(r, \frac{1}{g}) + 2N(r, \frac{1}{(g-1)m}) + 2N(r, \frac{1}{\prod_{j=1}^d g(z + c_j)^{s_j}})
\]

\[
+ S(r, f) + S(r, g)
\]

\[
(n + m + \sigma)[T(r, f) + T(r, g)] \leq (4 + 2m + 2\sigma)[T(r, f) + T(r, g)] + O(r^{\rho(f)-1+\epsilon})
\]

\[
+ O(r^{\rho(g)-1+\epsilon}) + S(r, f) + S(r, g)
\]

(4.5)

\[
(n - \sigma - m - 4)[T(r, f) + T(r, g)] \leq O(r^{\rho(f)-1+\epsilon}) + O(r^{\rho(g)-1+\epsilon}) + S(r, f) + S(r, g)
\]

which contradicts with $n \geq \sigma + m + 5$. Thus we have $H \equiv 0$. Note that

\[
\overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{G}) \leq (1 + m + \sigma)T(r, f) + (1 + m + \sigma)T(r, g) + S(r, f) + S(r, g) \leq T(r).
\]

where $T(r) = \max\{T(r, F), T(r, G)\}$. By Lemma 3.3, we deduce that either $F \equiv G$ or $FG \equiv 1$. Next we will consider the following two cases, respectively.

Let $FG = 1$. Then

\[
[f^n(z)(f(z)-1)^m \prod_{j=1}^d f(z + c_j)^{s_j}][g^n(z)(g(z)-1)^m \prod_{j=1}^d g(z + c_j)^{s_j}] = \alpha^2
\]

\[
[f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+...+1) \prod_{j=1}^d f(z+c_j)^{s_j}][g^n(z)(g(z)-1)^m(g^{m-1}(z)+g^{m-2}(z)+...+1) \prod_{j=1}^d g(z+c_j)^{s_j}] = \alpha^2.
\]

It can be easily viewed from above that

\[
N(r, \frac{1}{f}) = S(r, f) \text{ and } N(r, \frac{1}{f-1}) = S(r, f)
\]

Thus,

\[
\delta(0, f) + \delta(1, f) + \delta(\infty, f) = 3, \text{ which is not possible. Therefore we must have } F \equiv G.
\]

This completes the proof of Theorem 2.1.
Proof of Theorem 2.2. Let
\[ F(z) = \frac{[f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}]}{\alpha(z)}, \quad G(z) = \frac{[g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j}]}{\alpha(z)}. \]
Then \( F(z) \) and \( G(z) \) share \((1, 2)\) except the zeros or poles of \( \alpha(z) \). Obviously
\[
T(r, F) + T(r, G) \leq 2N_2(r, \frac{1}{F}) + 2N_2(r, \frac{1}{G}) + \mathcal{N}(r, \frac{1}{F}) + \mathcal{N}(r, \frac{1}{G}) + S(r, F) + S(r, G)
\]
\[
(n + m + \sigma)[T(r, f) + T(r, g)] \leq (5 + 3m + 3\sigma)[T(r, f) + T(r, g)] + O(r^{\rho(f)-1+\epsilon})
\]
\[
+ O(r^{\rho(g)-1+\epsilon}) + S(r, f) + S(r, g).
\]
(4.6)
\[
(n - 2m - 2\sigma - 5)[T(r, f) + T(r, g)] \leq O(r^{\rho(f)-1+\epsilon}) + O(r^{\rho(g)-1+\epsilon}) + S(r, f) + S(r, g).
\]
According to (4.6) and Lemma 3.2, we can prove Theorem 2.2 in a similar way as in proof of Theorem 2.1.

Proof of Theorem 2.3. Let
\[ F(z) = \frac{f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j}}{\alpha(z)}, \quad G(z) = \frac{g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j}}{\alpha(z)}. \]
Then \( \overline{E}_2 \left( 1, f^n(z)(f(z) - 1)^m \prod_{j=1}^{d} f(z + c_j)^{s_j} \right) = \overline{E}_2 \left( 1, g^n(z)(g(z) - 1)^m \prod_{j=1}^{d} g(z + c_j)^{s_j} \right) \) except the zeros or poles of \( \alpha(z) \).
Obviously
\[
T(r, F) + T(r, G) \leq 2N_2(r, \frac{1}{F}) + 2N_2(r, \frac{1}{G}) + 3\mathcal{N}(r, \frac{1}{F}) + 3\mathcal{N}(r, \frac{1}{G})
\]
\[
+ S(r, F) + S(r, G)
\]
\[
(n + m + \sigma)[T(r, f) + T(r, g)] \leq (7 + 5m + 5\sigma)[T(r, f) + T(r, g)] + O(r^{\rho(f)-1+\epsilon})
\]
\[
+ O(r^{\rho(g)-1+\epsilon}) + S(r, f) + S(r, g)
\]
(4.7)
\[
(n - 4m - 4\sigma - 7)[T(r, f) + T(r, g)] \leq O(r^{\rho(f)-1+\epsilon}) + O(r^{\rho(g)-1+\epsilon})S(r, f) + S(r, g).
\]
Using to (4.7) and Lemma 3.4, we can prove Theorem 2.3 in a similar way as in proof of Theorem 2.1.
5. Open Questions

Question 5.1 Can the Theorem 2.1 - 2.3 be extend to meromorphic functions?

Question 5.2 Can the difference polynomials in Theorem 2.1 - 2.3 be replaced by difference polynomials of the form $f^n(z)(f(z) - 1)^m \prod_{j=1}^d f(z + c_j)^{s_j} \prod_{j=1}^k f^{(i)}(z)$?

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