Kelvin-Helmholtz Instability of Cylindrical Geometry for Micro-dimensional Range of Wavelengths

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Abstract: The goal of this research is to determine input parameters necessary for a micro-dimensional range of wavelengths of disturbances. Therefore, appropriate parameters, furthering generation of a micrometer range in wavelengths of disturbances can be determined with the help of a numerical solution. A simplified dispersion equation is given for shortwave disturbances on the boundary of two viscous-potential liquids in a cylindrical geometry; the ratio of decrement to wavenumber in shortwave range can be determined with the help of this equation. The study has established that this ratio has two maximums for the iron/argon system. The first maximum falls within the millimeter range wavelength, whereas the second maximum is registered in a micrometer range. Speeds of liquid and gas are determined, which provide a micrometer range of wavelength of disturbances on the surface of liquid.

Keywords: Kelvin-Helmholtz instability, Cylindrical geometry, Wavelengths, Mathematical modeling.

PACS: 61.20.Ja, 61.20.Gy, 47.15.Fe, 47.15.Rq, 51.50.+V.

Introduction

Decaying of a liquid jet into individual drops caused by concentric circular gas flows is a quite universal phenomenon both in industry processes and in natural conditions [1-4]. Drops are formed, primarily, as a result of initiation and propagation of disturbances associated with a certain type of instability on the surface of a liquid jet. The resulting corrugated surface with increasing amplitude is a necessary condition for the formation of drops. Therefore, instability of a liquid jet in a gas flow – Kelvin-Helmholtz instability (KHI) in cylindrical geometry is an important issue. The wavelength of these disturbances is critical for the dimensions of drops. To determine conditions furthering generation of nano- and micro-dimensional drops, our study is focused on mechanisms explaining the initiation of instabilities.

Rayleigh’s research is a classical study on decaying of ideal non-viscous liquid jets into individual drops [5-7]. In these studies, a ratio of the growth rate of disturbances to a wavenumber has been suggested. He concluded that a wavelength with a maximum growth rate is \( \lambda = \)
4.508 \cdot 2R$, where $R$ is the radius of a liquid jet. Generation of micro- or nano-dimensional drops requires jets with a diameter of 10 nm, which represents a technological challenge.

Weber’s works considering viscosity of a liquid and omitting a gas flow are a further contribution to the theory of decay [8-10]. Squire [11-12] analyzed stability of a liquid, taking into consideration a gas flow, but viscosity was outside the scope of his work.

A model of KHI for a flat geometry is suggested in [13]; here, viscosity in a jet is introduced and a ratio of decrement vs. wavelength with two maxima (wavelengths $\lambda_{m1}$ and $\lambda_{m2}$) is derived on the base of dispersion equation solved numerically. The first one – aero-dynamical ($\lambda_{m1}$) – doesn’t depend on viscosity, whereas the second one ($\lambda_{m2}$) is associated with viscosity and $\lambda_{m1} > \lambda_{m2}$. Parameters of KHI for a plane geometry in a double-layer system were also determined [13]. In an ideally viscous liquid, maxima of decrement are in micro- and nano-ranges of wavelengths. Here, researchers relied on a viscous-potential model developed in [14]. This model emphasizes that effects caused by viscosity are registered only at the solid-liquid boundary. In this case, the Navier-Stokes equation is written for a perfect liquid, taking viscosity in boundary conditions into consideration. The study in [6] is particularly relevant, since the second maximum is in a nano-dimensional range of wavelengths, being possible in conditions with a relative sliding speed of around 50 m/s, whereas the only maximum registered in a nano-dimensional range is at speeds of about 1 km/s.

Researchers in [15] attempted to analyze the linear stability of two viscous liquids in cylindrical configuration under the assumption of their viscous-potential flow. As a result, neutral curves were obtained, which separate areas of stability and instability of a liquid flow. Studies in [16-19] have determined approximate analytical expressions for $\lambda_{m1}$ and $\lambda_{m2}$ for a plane viscous-potential flow. These studies have shown that ratios above are usable for the description of experimental data on the effect of heterogeneous plasma flows on metallic materials.

The objective of this research is to determine input parameters (liquid and gas speeds, radii of a liquid column and a gas medium) necessary for a micro-dimensional range of wavelengths of disturbances.

**Problem Statement**

We examine an initial stage of instability in the cylindrical column of a liquid with a density $\rho_1$ and a dynamic viscosity $\mu_1$, surrounded by a gas medium with a density $\rho_2$ and a dynamic viscosity $\mu_2$. The liquid takes up the area $R_1 < r < R$, $0 < \varphi < 2\pi$, $-\infty < z < \infty$, while the gas takes up the area $-R < r < R_2$, $0 < \varphi < 2\pi$, $-\infty < z < \infty$ (Fig. 1).

![FIG. 1. On problem statement of KHI in a cylindrical geometry.](image)

An undisturbed flow is set by non-zero axial speeds $w_{01}$, $w_{02}$; radial speed components of the undisturbed flow are zero and an undisturbed pressure is constant in both media. Therefore, effects caused by dynamic viscosity are possible only on the liquid-gas boundary. In this case, equations of motion, discontinuity and kinematic boundary conditions are equally satisfied. Boundary conditions for tangential components of stresses are irrelevant in a viscous-potential model. Both media are assumed to be incompressible. Navier-Stokes equations for disturbances of flow speeds in cylindrical coordinates are written as [20]:

\[
\begin{align*}
\frac{\partial u_z}{\partial z} + \frac{u_z}{r} \frac{\partial u_z}{\partial r} + \frac{\partial w_1}{\partial t} + w_1 \frac{\partial u_z}{\partial z} + \frac{\partial p_1}{\rho_1 \partial r} &= 0, \\
\frac{\partial w_1}{\partial t} + w_{10} \frac{\partial w_1}{\partial z} + \frac{\partial p_1}{\rho_1 \partial z} &= 0; \\
\frac{\partial u_2}{\partial t} + w_{20} \frac{\partial u_2}{\partial z} + \frac{\partial p_2}{\rho_2 \partial z} &= 0, \\
\frac{\partial w_2}{\partial t} + w_{20} \frac{\partial w_2}{\partial z} + \frac{\partial p_2}{\rho_2 \partial r} &= 0
\end{align*}
\]

(1)
Kinematic conditions on the boundary are stated $r = R + \eta(t, z)$:

$$\frac{\partial \eta}{\partial t} + w_0 \frac{\partial \eta}{\partial z} = u_1, \quad \frac{\partial \eta}{\partial t} + w_0 \frac{\partial \eta}{\partial z} = u_2. \quad (2)$$

Dynamic boundary conditions are written as:

$$p_2 - p_1 + 2\rho_1 \frac{\partial u_1}{\partial r} - 2\rho_2 \frac{\partial u_2}{\partial r} = \sigma \left( \frac{\eta}{R^2} + \frac{\partial \eta}{\partial r} \right). \quad (3)$$

On the outlines, impermeability conditions are set:

$$u_1(R_1) = 0, u_2(R_2) = 0 \quad (4)$$

A solution of (1) – (4) is written as follows:

$$u_1(r, z, t) = U_1(r) \exp(\epsilon t + k z), \quad u_2(r, z, t) = U_2(r) \exp(\epsilon t + k z),$$

$$w_1(r, z, t) = W_1(r) \exp(\epsilon t + k z), \quad w_2(r, z, t) = W_2(r) \exp(\epsilon t + k z).$$

$$p_1(r, z, t) = P_1(r) \exp(\epsilon t + k z), \quad p_2(r, z, t) = P_2(r) \exp(\epsilon t + k z).$$

$$\eta(z, t) = \eta_0 \exp(\epsilon t + k z). \quad (5)$$

Substituting (5) in (1), two equations are stated:

$$U^n + U' n / r - (k^2 - 1 / r^2) U_0 = 0; \quad n = 1, 2. \quad (6)$$

Here, formulae for pressure are written as:

$$P_1 = -\frac{\rho_1 (r \eta', \eta') + \eta_0}{r k^2} \Omega_1, \quad P_2 = -\frac{\rho_2 (r \eta', \eta') + \eta_0}{r k^2} \Omega_2, \quad (7)$$

where $\Omega_1 = \omega + ik w_{10}, \Omega_2 = \omega + ik w_{20}$. The expression below is written on the base of kinematic conditions:

$$\frac{u_1(R)}{\Omega_1} - \frac{u_2(R)}{\Omega_2} = 0. \quad (8)$$

Taking into account (4), (7) and (8) for each area, a solution is stated as follows:

$$U_1(r) = \frac{\eta_0 \Omega_2 F_1(R)}{\Omega_1}, \quad U_2(r) = \eta_0 \Omega_2 F_2(r),$$

$$F_n(r) = K_{1,2} \frac{\Omega_1(K_{1,2}^n) - I_{1,2}(K_{1,2}^n)}{\Omega_1 K_{1,2}^n - I_{1,2}(K_{1,2}^n)}. \quad (9)$$

We use (3), (7) and (9) to state the dispersion equation:

$$(E_2 \Omega_2^2 + 2\omega v_2 \Omega_2 (1 + E_2)) \theta - (E_1 \Omega_1^2 + 2\omega v_1 \Omega_1 (1 + E_1)) + \frac{x^2 - 1}{x} \omega^2 = 0,$$

where:

$$E_n = (-1)^n \left[ K_1(x_n) I_0(x) + K_0(x) I_1(x) \right].$$

Finally, a decrement for disturbances appears as:

$$\alpha = \sqrt{\frac{a^2 - b^2 - c + \sqrt{(a^2 - b^2 - c)^2 + (2ab - c_1)}}{2} - a} \quad (17)$$
For shortwaves, it is true that \((k>>1)\); so, \((x, x_y, x_z)>>1\) is also satisfied, and Bessel function is written as:
\[
I_0(x) \approx I_1(x) = \frac{\exp(x)}{\sqrt{2\pi x}}, \quad K_0(x) \approx K_1(x) = \frac{\pi \exp(-x)}{\sqrt{2\pi x}}.
\]
\[
\pi \exp(-x) \, K_1(x) \approx \pi \exp(-2x).
\]  
Then,
\[
E_1 \approx x \coth(k(R_1-R)), \quad E_2 \approx x \coth(k(R_2-R)).
\]  
Therefore, it’s possible to say that the cylindrical geometry transforms into a plane one in the case of shortwaves.

Fig. 2 shows the ratio of decrement of disturbances to wavenumber of cylindrical (curve 1) and plane (curve 2) cases on the example of water/air system. As seen, a maximum of decrement is higher for a plane case (1754 s\(^{-1}\)) than for a cylindrical one (1610 c\(^{-1}\)). A wavenumber (wavelength) with maximum is 4691.61 m\(^{-1}\) (\(\lambda =1.34\) mm) in plane geometry, while it is 4626.63 m\(^{-1}\) (\(\lambda =1.35\) mm) in cylindrical one. Therefore, approximation of plane geometry of a jet for shortwaves can be applied to the description of experimental data. So, to determine parameters required for macro- and nano-dimensional wavelengths, it is possible to use simplified approximated ratios of decrement to wavenumber introduced in [S].

**Results and Discussion**

For the purpose of modeling, we used pairs of materials (water / air, glycerin / air, iron / argon). Their characteristics are given in Table 1.

**TABLE 1. Characteristics of materials.**

<table>
<thead>
<tr>
<th>Material</th>
<th>(\rho), kg/m(^3)</th>
<th>(\mu), Pa·s</th>
<th>(\sigma), N/m</th>
</tr>
</thead>
<tbody>
<tr>
<td>water</td>
<td>997</td>
<td>8.94 \cdot 10(^{-4})</td>
<td>0.059</td>
</tr>
<tr>
<td>glycerin</td>
<td>1260</td>
<td>1.48</td>
<td>0.0647</td>
</tr>
<tr>
<td>iron</td>
<td>6700</td>
<td>4.4 \cdot 10(^{-3})</td>
<td>1.2</td>
</tr>
<tr>
<td>air</td>
<td>1.1308</td>
<td>1.7798 \cdot 10(^{-5})</td>
<td>-</td>
</tr>
<tr>
<td>argon</td>
<td>0.2434</td>
<td>8.07 \cdot 10(^{-5})</td>
<td>-</td>
</tr>
</tbody>
</table>

Fig. 3 gives the ratios of decrement to wavenumber for water / air system at \(R = 2.5\) mm, \(R_2 = 12.5\) mm, \(w_1 = 0.28\) m/s, \(w_2 = 19.5\) m/s (curve 1) and \(R = 2.5\) mm, \(R_3 = 7.5\) mm, \(w_1 = 0.28\) m/s, \(w_2 = 15.2\) m/s (curve 2). As seen, wavenumber with the maximum ratio is \(k_m = 4627.13\) m\(^{-1}\) and wavelength is 1.36 mm; in the second case, \(k_m = 2780.54\) m\(^{-1}\) and \(\lambda_m = 2.26\) mm. Maxima of decrement are \(a_{m1} = 1611\) c\(^{-1}\) and \(a_{m2} = 724\) c\(^{-1}\).

![FIG. 2. Decrement of disturbances vs. wavenumber for cylindrical (curve 1) and plane (curve 2) cases on the example of water / air system.](image)

![FIG. 3. Decrement of disturbances on the liquid-gas boundary vs. wavenumber for water / air system. 1) R = 2.5 mm, R_1 = 12.5 mm, w_1 = 0.28 m/s, w_2 = 19.5 m/s; 2) R = 2.5 mm, R_3 = 7.5 mm, w_1 = 0.28 m/s, w_2 = 15.2 m/s.](image)

For glycerin/air system, the ratio of decrement of disturbance on the liquid-gas boundary to wavenumber is shown in Fig. 4 at \(R = 0.5\) mm, \(R_2 = 10\) mm, \(w_1 = 10\) m/s, \(w_2 = 60\) m/s and \(R = 1\) mm, \(R_3 = 10\) mm, \(w_1 = 10\) m/s, \(w_2 = 110\) m/s. The wavenumber in the first case is \(k_m = 5782.14\) m\(^{-1}\), so the wavelength is 1.08 mm. In the second case, \(k_m = 8535.14\) m\(^{-1}\) and \(\lambda_m = 0.74\) mm. Maxima of decrement are \(a_{m1} = 778\) s\(^{-1}\) and \(a_{m2} = 3538\) s\(^{-1}\).
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FIG. 4. Decrement of disturbances on the liquid-gas boundary vs. wavenumber for glycerin / air system. 1) $R = 0.5\, \text{mm}, R_2 = 10\, \text{mm}, w_1 = 10\, \text{m/s}, w_2 = 60\, \text{m/s};$ 2) $R = 1\, \text{mm}, R_2 = 10\, \text{mm}, w_1 = 10\, \text{m/s}, w_2 = 110\, \text{m/s}.$

Fig. 5 shows the ratio of decrement of disturbances on the liquid-gas boundary vs. wavenumber for liquid iron/argon system. This function has two maxima for micro- and millimeter range of wavelengths. At a gas speed of 100 m/s (Fig. 4, curve 1), the wavelength with the first maximum is 4.22 mm ($k_{m1} = 1488\, \text{m}^{-1}$). The second maximum is at the wavelength of 0.47 mm ($k_{m2} = 13482\, \text{m}^{-1}$). Maxima of decrement are 405 s$^{-1}$ and 50 s$^{-1}$. This is an evidence on splitting of a jet into two kinds of drops. Increasing the speed of gas to 120 m/s, wavelengths are shortened to 3.17 mm and 0.33 mm, respectively. In this case, maxima of decrement are 640 s$^{-1}$ and 98 s$^{-1}$. Tables 2 – 4 provide data on wavelengths with maxima of decrement according to speeds of liquid and gas, and radii of jets. As seen, as the speed of media motion increases, the waves become shorter and the maximal decrement becomes higher.

TABLE 2. Wavelength with maximum decrement vs. initial parameters of experiment for water / air system.

<table>
<thead>
<tr>
<th>Speed of liquid jet, m/s</th>
<th>Speed of gas jet, m/s</th>
<th>Diameter of liquid jet, mm</th>
<th>Diameter of gas jet, mm</th>
<th>$\lambda_{m}$, mm</th>
<th>$\alpha_{m}$, s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.4</td>
<td>31</td>
<td>5</td>
<td>25</td>
<td>0.57</td>
<td>6104</td>
</tr>
<tr>
<td>0.28</td>
<td>19.5</td>
<td>5</td>
<td>25</td>
<td>1.36</td>
<td>1611</td>
</tr>
<tr>
<td>0.28</td>
<td>15.2</td>
<td>5</td>
<td>15</td>
<td>2.26</td>
<td>724</td>
</tr>
<tr>
<td>0.28</td>
<td>37</td>
<td>5</td>
<td>15</td>
<td>0.37</td>
<td>11744</td>
</tr>
<tr>
<td>0.28</td>
<td>60</td>
<td>5</td>
<td>15</td>
<td>0.07</td>
<td>50571</td>
</tr>
</tbody>
</table>

TABLE 3. Wavelength with maximum decrement vs. initial parameters of experiment for glycerin / air system.

<table>
<thead>
<tr>
<th>Speed of liquid jet, m/s</th>
<th>Speed of gas jet, m/s</th>
<th>Diameter of liquid jet, mm</th>
<th>Diameter of gas jet, mm</th>
<th>$\lambda_{m}$, mm</th>
<th>$\alpha_{m}$, s$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>60</td>
<td>0.5</td>
<td>20</td>
<td>0.72</td>
<td>689.9</td>
</tr>
<tr>
<td>10</td>
<td>110</td>
<td>0.5</td>
<td>20</td>
<td>0.44</td>
<td>3384</td>
</tr>
<tr>
<td>10</td>
<td>160</td>
<td>0.5</td>
<td>20</td>
<td>0.32</td>
<td>7995</td>
</tr>
<tr>
<td>10</td>
<td>210</td>
<td>0.5</td>
<td>20</td>
<td>0.25</td>
<td>14487</td>
</tr>
<tr>
<td>10</td>
<td>310</td>
<td>0.5</td>
<td>20</td>
<td>0.16</td>
<td>33125</td>
</tr>
<tr>
<td>10</td>
<td>510</td>
<td>0.5</td>
<td>20</td>
<td>0.087</td>
<td>93122</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>1</td>
<td>20</td>
<td>1.08</td>
<td>777.8</td>
</tr>
<tr>
<td>Speed of liquid jet, m/s</td>
<td>Speed of gas jet, m/s</td>
<td>Diameter of liquid jet, mm</td>
<td>Diameter of gas jet, mm</td>
<td>$\lambda_{m1}$, mm</td>
<td>$\alpha_{m1}$, $c^{-1}$</td>
</tr>
<tr>
<td>-------------------------</td>
<td>----------------------</td>
<td>---------------------------</td>
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<td>10</td>
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<td>2</td>
<td>20</td>
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<tr>
<td>10</td>
<td>210</td>
<td>2</td>
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<td>0.30</td>
<td>14594</td>
</tr>
</tbody>
</table>

TABLE 4. Wavelength with maximum decrement vs. initial parameters of experiment for iron / argon system.

Therefore, appropriate parameters (speeds of liquid and gas and radii of liquid column and gas flow) for furthering generation of a micrometer range in wavelengths of disturbances can be determined with the help of numerical solution (17). For water / air system, a micrometer range is possible at gas speeds of $w_{02} > 60$ m/s and liquid speeds of $w_{01} < 1$ m/s. The diameter of air column is to be fivefold greater than the diameter of liquid column. Investigating KHI in iron / argon system, the conclusion is made that speeds of gas $w_{02} > 100$ m/s and liquid $w_{01} \approx 1$ m/s are required for this range. Here, the diameter of the liquid jet is to be significantly smaller than the diameter of the air column. Since dynamic viscosity of glycerin is high, a speed of air of $w_{02} > 500$ m/s is required to obtain a micrometer range of wavelengths.

Conclusions

A simplified dispersion equation is stated for shortwave disturbances on the boundary of two viscous-potential liquids in cylindrical geometry. The ratio of decrement to wavenumber in shortwave range can be determined with the help of this equation. Ratios of decrement (growth rate) to wavenumber are found and analyzed on the example of water / air, glycerin / air and iron / argon systems. The study has established that this ratio has two maxima for iron / argon system. The wavelength with the first maximum is in the millimeter range, whereas the second maximum registered is in the micrometer range. Speeds of liquid and gas are determined which provide a micrometer range of wavelength of disturbances on the surface of liquid. The study has shown that this range is possible, provided that the diameter of the jet is substantially smaller than the diameter of the air column. A comparative analysis of computational and experimental data is required to prove the adequacy of the research outcomes.

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