ON Λ-GENERALIZED CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

K. BALASUBRAMANIYAN (1), S. SRIRAM (2) AND O. RAVI (3)

Abstract. In this paper, we introduce the concepts of Λ-generalized fuzzy closed sets (briefly, Λgf-closed sets), Λ-gf-closed sets and gf-Λ-closed sets in fuzzy topological spaces. Also we study some properties and characterizations of Λ-generalized fuzzy closed sets.

1. Introduction

In 1986, Maki [11] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel (=saturated set), i.e. the intersection of all open supersets of A. Arenas et al. [1] introduced and investigated the notion of λ-closed sets by involving Λ-sets and closed sets. A subset A of a topological space (X, τ) is called λ-closed [1] if A = L ∩ D, where L is a Λ-set and D is a closed set. The intersection of all λ-closed sets containing a subset A of X is called the λ-closure of A and is denoted by cl_λ(A) [5]. The complement of a λ-closed set is called λ-open. Ganster and Reilly [7] introduced the notion of locally closed sets using open sets and closed sets. In 1970, Levine [10] introduced the notion of generalized closed sets (briefly, g-closed sets) in topological spaces as a generalization of closed sets. Since
then, many concepts related to generalized closed sets were defined and investigated. Caldas et al. [4] introduced new classes of sets called $\Lambda_g$-closed sets and $\Lambda_g$-open sets in topological spaces. They also established several properties of such sets. It is proved that $\Lambda_g$-closed sets and $\Lambda_g$-open sets are weaker forms of closed sets and open sets, respectively and stronger forms of $g$-closed sets and $g$-open sets, respectively.

Since the generalization of the usual notion of a set into a fuzzy set by Zadeh in his classic paper [17] of 1965, many abstract structures were generalized using fuzzy sets. Fuzzy topological spaces were introduced by Chang [6]. Fuzzy continuous functions and fuzzy closed functions were introduced by Chang in [6]. Recently Balasubramanian and Sundaram [3] introduced and studied the concepts of generalized fuzzy closed sets and fuzzy $T_{1/2}$-spaces in fuzzy topological spaces. Moreover, they studied the generalizations of fuzzy continuous functions.

In the present paper, we introduce the concepts of $\Lambda$-generalized fuzzy closed sets (briefly, $\Lambda_{gf}$-closed sets), $\Lambda$-gf-closed sets and $gf$-$\Lambda$-closed sets in fuzzy topological spaces. Further, we study some properties and characterizations of $\Lambda$-generalized fuzzy closed sets. Suitable Examples are given at proper places to substantiate the results. In topological spaces, the symbols such as $\subseteq$, $\cap$ and $\cup$ are used. Correspondingly, $\leq$, $\wedge$ and $\vee$ symbols are used in fuzzy topological spaces.

2. Preliminaries

A map from a nonempty set $X$ into the closed unit interval $I = [0, 1]$ is a fuzzy subset of $X$. The constant fuzzy sets taking the values 0 and 1 on $X$ are denoted by $0_X$ and $1_X$ respectively. The family of all fuzzy sets of $X$ is denoted by $I_X$. Usually the fuzzy sets will be denoted by Greek letters such as $\mu$, $\rho$, $\nu$, $\lambda$, $\alpha$, $\beta$, .... or English alphabets such as $A$, $B$, $C$, ...

**Definition 2.1.** [6] A family $\tau$ of fuzzy sets on $X$ is called a fuzzy topology for $X$ if

1. $0_X, 1_X \in \tau$, (2) $\mu \wedge \rho \in \tau$ whenever $\mu, \rho \in \tau$ and (3) $\vee\{\mu_i : i \in \Delta\} \in \tau$ whenever each $\mu_i \in \tau (i \in \Delta)$.
Moreover, the pair \((X, \tau)\) is called a fuzzy topological space.
Every member of \(\tau\) is called a fuzzy open set.
The complement of a fuzzy open set is called a fuzzy closed set.
The complement of a fuzzy set \(\lambda\) of \(X\) is \(1-\lambda\) (or \(\lambda^1\)).

**Definition 2.2.** For a fuzzy set \(\lambda\) of \((X, \tau)\), the closure \(cl(\lambda)\) and the interior \(int(\lambda)\) of \(\lambda\) are defined in [2] respectively, as
\[
cl(\lambda) = \bigwedge\{\nu : \nu \geq \lambda, \nu^1 \in \tau\} \quad \text{and} \\
int(\lambda) = \bigvee\{\nu : \nu \leq \lambda, \nu \in \tau\}.
\]

**Definition 2.3.** [2] Let \((X, \tau)\) be a fuzzy topological space. A fuzzy set \(\mu\) of \(X\) is called
- (1) fuzzy regular open if \(\mu = int(cl(\mu))\);
- (2) fuzzy regular closed if \(\mu = cl(int(\mu))\);
It is easily seen that a fuzzy set \(\mu\) is fuzzy regular open if and only if \(\mu^1\) is fuzzy regular closed.

**Definition 2.4.** [3] A fuzzy set \(\mu\) of a fuzzy topological space \((X, \tau)\) is called generalized fuzzy closed (briefly, gf-closed) if \(cl(\mu) \leq \lambda\) whenever \(\mu \leq \lambda\) and \(\lambda \in \tau\).

**Definition 2.5.** [14] A fuzzy set \(\mu\) of a fuzzy topological space \((X, \tau)\) is called a fuzzy LC set if \(\mu = \alpha \land \beta\) where \(\alpha\) is a fuzzy open and \(\beta\) is a fuzzy closed.

**Definition 2.6.** [3] A fuzzy topological space \((X, \tau)\) is called fuzzy \(T_{1/2}\) space if every gf-closed set is fuzzy closed.

**Definition 2.7.** Let \(\mu\) be a fuzzy set of a fuzzy topological space \((X, \tau)\). Then \(\mu\) is said to be
- (1) fuzzy semiopen if and only if \(\mu \leq cl(int(\mu))\) [2];
- (2) fuzzy semiclosed if and only if \(\mu^1\) is a fuzzy semiopen set of \(X\) [2];
- (3) fuzzy preopen if and only if \(\mu \leq int(cl(\mu))\) [15];
- (4) fuzzy preclosed if and only if \(\mu^1\) is a fuzzy preopen set of \(X\) [15].
Definition 2.8. Let $\mu$ be a fuzzy set of a fuzzy topological space $(X, \tau)$. Then

1. $\text{pint}(\mu) = \bigvee \{\lambda \mid \lambda \leq \mu, \lambda \text{ is a fuzzy preopen set of } X\}$, is called the fuzzy preinterior of $\mu$ [8];
2. $\text{pcl}(\mu) = \bigwedge \{\lambda \mid \lambda \geq \mu, \lambda \text{ is a fuzzy preclosed set of } X\}$, is called the fuzzy preclosure of $\mu$ [8];
3. $\text{sint}(\mu) = \bigvee \{\lambda \mid \lambda \leq \mu, \lambda \text{ is a fuzzy semiopen set of } X\}$, is called the fuzzy semiinterior of $\mu$ [16];
4. $\text{scl}(\mu) = \bigwedge \{\lambda \mid \lambda \geq \mu, \lambda \text{ is a fuzzy semiclosed set of } X\}$, is called the fuzzy semiclosure of $\mu$ [16].

Theorem 2.9. Let $\mu$ be a fuzzy set in a fuzzy topological space $(X, \tau)$. Then

1. $\text{pint}(\mu) \leq \mu \land \text{int}(\text{cl}(\mu))$ [8];
2. $\text{pcl}(\mu) \geq \mu \lor \text{cl}(\text{int}(\mu))$ [8];
3. $\text{sint}(\mu) = \mu \land \text{cl}(\text{int}(\mu))$ [9];
4. $\text{scl}(\mu) = \mu \lor \text{int}(\text{cl}(\mu))$ [9].

The following Lemma and two definitions are introduced in [6].

Definition 2.10. Let $f : X \rightarrow Y$ be a function from a set $X$ into a set $Y$, $\mu$ be a fuzzy subset in $X$ and $\rho$ be a fuzzy subset in $Y$. Then the Zadeh’s functions $f(\mu)$ and $f^{-1}(\rho)$ are defined by

1. $f(\mu)$ is a fuzzy subset of $Y$ where
   $$f(\mu) = \begin{cases} \sup_{z \in f^{-1}(y)} \mu(z) & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$
   for each $y \in Y$.
2. $f^{-1}(\rho)$ is a fuzzy subset of $X$ where $f^{-1}(\rho)(x) = \rho(f(x))$ for each $x \in X$.
3. $(f^{-1}(\rho))^1 = f^{-1}(\rho^1)$.

Lemma 2.11. Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then for fuzzy sets $\mu$ and $\rho$ of $X$ and $Y$ respectively, the following statements hold:
(1) $ff^{-1}(\rho) \leq \rho$;
(2) $f^{-1}f(\mu) \geq \mu$;
(3) $f(\mu^1) \geq (f(\mu))^1$;
(4) $f^{-1}(\rho^1) = (f^{-1}(\rho))^1$;
(5) if $f$ is injective, then $f^{-1}(f(\mu)) = \mu$;
(6) if $f$ is surjective, then $ff^{-1}(\rho) = \rho$;
(7) if $f$ is bijective, then $f(\mu^1) = (f(\mu))^1$.

Definition 2.12. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is called

(1) fuzzy continuous if the inverse image of each fuzzy open set of $Y$ is fuzzy open in $X$,
(2) fuzzy closed if the image of each fuzzy closed set of $X$ is fuzzy closed in $Y$.

Definition 2.13. [13] Let $(X, \tau)$ be a fuzzy topological space. A fuzzy point $x_\alpha$ ($0 < \alpha \leq 1$) is a fuzzy set of $X$ defined as follows:

$$x_\alpha(y) = \begin{cases} \alpha & \text{if } y = x; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.14. ([12], Definition 1.2.4) Let $(X, \tau)$ be a fuzzy topological space. The fuzzy point $x_t$ in $X$ is said to be contained in a fuzzy set $\mu$ or to belong to $\mu$, denoted by $x_t \in \mu$, if and only if $t \leq \mu(x)$, for each $x \in X$. Evidently every fuzzy set $\mu$ can be expressed as the union of all the fuzzy points which belong to $\mu$.

Definition 2.15. ([12], Definition 8.2.3) A fuzzy topological space $(X, \tau)$ is called a $FT_1$ space if and only if every fuzzy point is a fuzzy closed set.

Definition 2.16. [13] Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $\mu$ is quasi-coincident with a fuzzy set $\nu$, denoted by $\mu q \nu$, if there exists $x \in X$ such that $\mu(x) + \nu(x) > 1$. If $\mu$ is not quasi-coincident with $\nu$, then we write $\mu q \nu$. It is known that $\mu \leq \nu$ if and only if $\mu q (1 - \nu)$.
Definition 2.17. [13] Let \((X, \tau)\) be a fuzzy topological space. A fuzzy point \(x_\alpha\) in \(X\) is quasi-coincident with a fuzzy set \(\nu\), denoted by \(x_\alpha \sim \nu\), if and only if \(\alpha + \nu(x) > 1\). If \(x_\alpha\) is not quasi-coincident with \(\nu\), then we write \(x_\alpha \not\sim \nu\).

Lemma 2.18. [2] For a fuzzy set \(\lambda\) of a fuzzy topological space \((X, \tau)\), the following hold:

1. \((\text{int}(\lambda))^1 = \text{cl}(\lambda^1)\);
2. \((\text{cl}(\lambda))^1 = \text{int}(\lambda^1)\).

3. \(\Lambda\)-generalized fuzzy closed sets

Definition 3.1. A fuzzy set \(\mu\) of a fuzzy topological space \((X, \tau)\) is called a \(\Lambda\)-fuzzy set if \(\mu = \hat{\mu}\), where \(\hat{\mu} = \land\{\lambda : \mu \leq \lambda, \lambda \in \tau\}\).

Remark 3.2. For a fuzzy set \(\mu\) of a fuzzy topological space \((X, \tau)\), the following properties hold:

1. \(\mu \leq \hat{\mu}\).
2. If \(\mu \in \tau\), then \(\hat{\mu} = \mu\) and hence is a \(\Lambda\)-fuzzy set.
3. If \(\hat{\mu} = \mu\), then the following Example 3.3 shows that \(\mu\) need not be fuzzy open.
4. If \(\mu \leq \sigma\), then \(\hat{\mu} \leq \hat{\sigma}\).
5. If \(\mu\) is any fuzzy subset of \(X\), then \(\hat{\mu} = \hat{\mu}\).

Proof. (1), (2), (3), (4) Obvious.

(5) If \(\rho \in \tau\) then \(\mu \leq \rho \iff \hat{\mu} \leq \rho\) by the definition of \(\hat{\mu}\) and (1). Hence \(\hat{\mu} = \hat{\mu}\) and \(\hat{\mu}\) is a \(\Lambda\)-fuzzy set.

Example 3.3. Let \(X\) be a nonempty set. Define \(C_a : X \rightarrow [0, 1]\) such that \(C_a(x) = a\) for all \(x \in X\) and \(a \in [0, 1]\). Then \(\tau = \{C_0, C_1, C_a : 3/10 < a \leq 4/10\}\) is a fuzzy topology on \(X\) and \((X, \tau)\) is a fuzzy topological space. Now \(\hat{C}_{3/10} = \land\{\rho : \rho \in \tau\} = C_b\) where \(b = \land\{a \mid 3/10 < a \leq 4/10\}\) and hence \(b = 3/10\) and \(\hat{C}_{3/10} = C_{3/10}\). Thus \(C_{3/10}\) is a \(\Lambda\)-fuzzy set but not fuzzy open in \(X\).
Proposition 3.4. In a FT$_1$ space $(X, \tau)$, every fuzzy subset of $X$ is a $\Lambda$-fuzzy set.

Proof. Let $\mu$ be any fuzzy subset of $X$ such that $x_t \mu$. Then $\mu(x) \leq 1 - t$. Since $(X, \tau)$ is a FT$_1$-space, $x_t$ is a fuzzy closed set of $X$. Hence $x^t_1$ is fuzzy open containing $\mu$. By definition of $\hat{\mu}$, $\hat{\mu} \leq x^t_1$ and therefore $x_t \hat{\mu}$. Thus $\mu(x) \leq 1 - t \Rightarrow \hat{\mu}(x) \leq 1 - t$ and hence $\hat{\mu} = \mu$. Then $\hat{\mu}$ is a $\Lambda$-fuzzy set.

Definition 3.5. A fuzzy set $\mu$ of a fuzzy topological space $(X, \tau)$ is said to be $\lambda$-fuzzy closed (briefly, $\lambda$-f-closed) if $\mu$ can be put in the form $\mu = \alpha \wedge \beta$ where $\alpha$ is a $\Lambda$-fuzzy set and $\beta$ is fuzzy closed in $X$.

The complement of a $\lambda$-f-closed set is $\lambda$-f-open. The collection of all $\lambda$-f-open sets in $(X, \tau)$ is denoted by $\lambda FO(X)$.

Lemma 3.6. In a fuzzy topological space $(X, \tau)$, the following properties hold:

1. If $\mu_i$ is $\lambda$-f-closed for each $i \in \Delta$, then $\bigwedge_{i \in \Delta} \mu_i$ is $\lambda$-f-closed.
2. If $\mu_i$ is $\lambda$-f-open for each $i \in \Delta$, then $\bigvee_{i \in \Delta} \mu_i$ is $\lambda$-f-open.
3. Intersection of two $\lambda$-f-open sets is not necessarily $\lambda$-f-open.

Proof. (1) Since $\mu_i$ is $\lambda$-f-closed, $\mu_i = \alpha_i \wedge \beta_i$ where $\alpha_i$ is a $\Lambda$-fuzzy set and $\beta_i$ is fuzzy closed for each $i$. Therefore $\bigwedge_{i \in \Delta} \mu_i = \bigwedge_{i \in \Delta} (\alpha_i \wedge \beta_i) = \bigwedge_{i \in \Delta} \alpha_i \wedge \bigwedge_{i \in \Delta} \beta_i$.

Now $\bigwedge_{i \in \Delta} \alpha_i \leq \alpha_i$ for each $i$ and by (4) of Remark 3.2. We have $[\bigwedge_{i \in \Delta} \alpha_i] \leq \alpha_i$ for each $i$ since each $\alpha_i$ is a $\Lambda$-fuzzy set.

Hence $[\bigwedge_{i \in \Delta} \alpha_i] \leq \bigwedge_{i \in \Delta} \alpha_i$ and thus $\bigwedge_{i \in \Delta} \alpha_i$ is a $\Lambda$-fuzzy set.

Since $\beta_i$ is fuzzy closed for each $i$, $\bigwedge_{i \in \Delta} \beta_i$ is fuzzy closed and hence $\bigwedge_{i \in \Delta} \mu_i$ is $\lambda$-f-closed.

(2) Taking complements the proof follows from (1).

(3) Let $X = \{a, b\}$ and $A_1 : X \to [0, 1]$ be defined as $A_1(a) = 0.4$ and $A_1(b) = 0.6$. Then $\tau = \{0_X, 1_X, A_1\}$ is a fuzzy topology on $X$ with $A_1 = \{(a, 0.4), (b, 0.6)\}$. In $(X, \tau)$, $A_1$ and $A_2 = A_1^1$ are $\lambda$-f-open subsets by (4) of Remark 3.7. $(A_1 \wedge A_2)^1$ is not $\lambda$-f-closed, since the only $\Lambda$-fuzzy sets of $X$ are $0_X$, $A_1$ and
1 and the only fuzzy closed sets of X are 0, A₂, 1. Thus A₁∩A₂ is not λf-open.

**Remark 3.7.** The following statements are true for any fuzzy topological space.

1. Every Λ-fuzzy set of X is λf-closed in X.
2. Every fuzzy closed set of X is λf-closed in X.
3. Every fuzzy LC set of X is λf-closed in X.
4. Every fuzzy open set of X is both λf-open and λf-closed.

**Proof.** (1) Let μ be a Λ-fuzzy set. Then μ = μ∧Iₓ where Iₓ is fuzzy closed of X. Hence μ is λf-closed in X.

(2) Let μ be a fuzzy closed set of X. Then μ = Iₓ∧μ where Iₓ is a Λ-fuzzy set of X. Hence μ is λf-closed in X.

(3) Let μ be a fuzzy LC set. Then μ = α∧β where α is a fuzzy open set and β is a fuzzy closed set of X. By (2) of Remark 3.2, α is a Λ-fuzzy set of X and hence μ is λf-closed in X.

(4) Let μ be a fuzzy open of X. By (2) of Remark 3.2 and (1) of Remark 3.7, μ is λf-closed. Again by (2) of Remark 3.7, μ is λf-open. Thus μ is both λf-closed and λf-open in X.

The converse of each statement in Remark 3.7 is not true can be shown by the following Example.

**Example 3.8.** (1) In Example 3.3, C₆/₁₀ is fuzzy closed since C₄/₁₀ ∈ τ. By (2) of Remark 3.7, C₆/₁₀ is λf-closed. But Ĉ₆/₁₀ = C₁ ≠ C₆/₁₀ and thus it is not a Λ-fuzzy set.

(2) In Example 3.3, C₃/₁₀ is a Λ-fuzzy set since Ĉ₃/₁₀ = C₃/₁₀ and hence by (1) of Remark 3.7, C₃/₁₀ is λf-closed but not fuzzy closed in X.

(3) In Example 3.3, C₃/₁₀ is λf-closed by (1) of Remark 3.7. But C₃/₁₀ is not a fuzzy LC set. If C₃/₁₀ = α∧β where α is fuzzy open and β is fuzzy closed in X, then C₃/₁₀ = C₃∩C₄. Since C₃ is fuzzy open, b > 3/₁₀ and C₄ is fuzzy closed.
implies \( d \geq 6/10 \). Thus \( 3/10 = \min\{b,d\} \) which is a contradiction, proving that \( C_{3/10} \) is not a fuzzy LC set of \( X \).

(4) In Example 3.3, \( C_{6/10} \) is \( \lambda f \)-closed by (2) of Remark 3.7. Since \( C_{4/10} \) is fuzzy open, by (4) of Remark 3.7, \( C_{4/10} \) is \( \lambda f \)-closed and hence \( C_{6/10} \) is \( \lambda f \)-open. Thus \( C_{6/10} \) is both \( \lambda f \)-closed and \( \lambda f \)-open but not fuzzy open.

**Remark 3.9.** From (1) and (2) of Example 3.8, it is easy to see that \( \Lambda \)-fuzzyness and fuzzy closedness are independent.

**Lemma 3.10.** For a fuzzy set \( \mu \) of a fuzzy topological space \((X, \tau)\), the following conditions are equivalent.

1. \( \mu \) is \( \lambda f \)-closed.
2. \( \mu = L \land \text{cl}(\mu) \) where \( L \) is a \( \Lambda \)-fuzzy set.
3. \( \mu = \hat{\mu} \land \text{cl}(\mu) \).

**Proof.** (1)\( \Rightarrow \) (2). Obvious since \( \text{cl}(\mu) \) is fuzzy closed containing \( \mu \).

(2)\( \Rightarrow \) (3). Obvious since \( \hat{\mu} \) is a \( \Lambda \)-fuzzy set, by (5) of Remark 3.2.

(3)\( \Rightarrow \) (1). Follows since \( \hat{\mu} \) is a \( \Lambda \)-fuzzy set.

**Lemma 3.11.** A fuzzy set \( \mu \) of a fuzzy topological space \((X, \tau)\) is gf-closed if and only if \( \text{cl}(\mu) \leq \hat{\mu} \).

**Proof.** Let \( \mu \) be gf-closed in \( X \). Then \( \text{cl}(\mu) \leq \lambda \), whenever \( \mu \leq \lambda \) for any \( \lambda \in \tau \). Thus \( \text{cl}(\mu) \leq \land \{\lambda : \mu \leq \lambda \text{ and } \lambda \in \tau\} = \hat{\mu} \).

Conversely. Let \( \mu \leq \lambda \) and \( \lambda \in \tau \). By the definition of \( \hat{\mu} \), \( \hat{\mu} \leq \lambda \). Then \( \text{cl}(\mu) \leq \hat{\mu} \leq \lambda \).

Thus \( \text{cl}(\mu) \leq \lambda \) whenever \( \mu \leq \lambda \) and \( \lambda \in \tau \), which proves that \( \mu \) is gf-closed.

**Theorem 3.12.** For a fuzzy set \( \mu \) of a fuzzy topological space \((X, \tau)\), the following conditions are equivalent.

1. \( \mu \) is fuzzy closed.
2. \( \mu \) is gf-closed and a fuzzy LC set.
3. \( \mu \) is gf-closed and \( \lambda f \)-closed.
Proof. (1)⇒(2). Since every fuzzy closed set is gf-closed and also a fuzzy LC set, the proof follows immediately.

(2)⇒(3). By (3) of Remark 3.7, every fuzzy LC set is λf-closed and hence the proof.

(3)⇒(1). Since μ is gf-closed, then by Lemma 3.11, cl(μ)≤μ. But μ is λf-closed, hence by Lemma 3.10, μ = μ∧cl(μ). Therfore μ = cl(μ) and thus μ is fuzzy closed.

Definition 3.13. For a fuzzy set μ of a fuzzy topological space (X, τ), clλ(μ) is defined as the intersection of all λf-closed sets containing μ and is called the λf-closure of μ.

Remark 3.14. The following properties hold in any fuzzy topological space:

(1) For a fuzzy set μ, μ≤clλ(μ).
(2) clλ(μ) is λf-closed for a fuzzy set μ.
(3) If μ is λf-closed, then μ = clλ(μ).
(4) If μ≤σ, then clλ(μ)≤clλ(σ).
(5) For a fuzzy set μ, clλ(μ)≤cl(μ).

Definition 3.15. A fuzzy set μ of a fuzzy topological space (X, τ) is called Λ-generalized fuzzy closed (briefly, Λgf-closed) (respectively, Λ-gf-closed, gf-Λ-closed) if cl(μ)≤β (respectively, clλ(μ)≤β, clλ(μ)≤β) whenever μ≤β and β is λf-open (respectively, β is λf-open, β is fuzzy open).

As a consequence of the above definition, we have the following Proposition.

Proposition 3.16. For a fuzzy topological space (X, τ), then the following properties hold.

(1) Every fuzzy closed set is Λgf-closed.
(2) Every Λgf-closed set is gf-closed.
(3) Every λf-closed set is Λ-gf-closed.
(4) Every Λ-gf-closed set is gf-Λ-closed.
(5) Every Λgf-closed set is Λ-gf-closed.
(6) Every gf-closed set is gf-Λ-closed.
Proof. (1) Let $\mu$ be a fuzzy closed set and $\lambda$ be any $\lambda f$-open set containing $\mu$. Then $\text{cl}(\mu) = \mu \leq \lambda$. Thus $\text{cl}(\mu) \leq \lambda$ and hence $\mu$ is $\Lambda \text{gf}$-closed.

(2) Since any fuzzy open set $\lambda$ is $\lambda f$-open by (2) of Remark 3.7, the proof follows.

(3) Proof follows from (3) of Remark 3.14.

(4) Proof follows from (2) of Remark 3.7.

(5) Proof follows from (5) of Remark 3.14.

(6) Proof follows from (5) of Remark 3.14.

Remark 3.17. From the above discussions, we have the following diagram:

$$
\begin{array}{ccc}
\text{fuzzy closed} & \longrightarrow & \Lambda \text{gf-closed} \\
\downarrow & & \downarrow \\
\lambda f\text{-closed} & \longrightarrow & \Lambda \text{-gf-closed} \\
\downarrow & & \downarrow \\
\Lambda \text{gf-closed} & \longrightarrow & \text{gf-}\Lambda\text{-closed}
\end{array}
$$

Remark 3.18. From the single Example 3.3 it is seen that none of the above implications is reversible.

The different types of fuzzy sets other than $C_0$ and $C_1$ in Example 3.3 are

- fuzzy open sets = $\{C_a : 3/10 < a \leq 4/10\}$;
- fuzzy closed sets = $\{C_a : 6/10 \leq a < 7/10\}$;
- $\Lambda$-fuzzy sets = $\{C_a : 3/10 \leq a \leq 4/10\}$;
- $\lambda f$-closed sets = $\{C_a : 3/10 \leq a \leq 4/10, 6/10 \leq a < 7/10\}$;
- $\lambda f$-open sets = $\{C_a : 3/10 < a \leq 4/10, 6/10 \leq a \leq 7/10\}$;
- gf-closed sets = $\{C_a : 4/10 < a \leq 1\}$;
- $\Lambda$-gf-closed sets = $\{C_a : 0 \leq a < 7/10, 7/10 < a \leq 1\}$;
- $\Lambda$gf-closed sets = $\{C_a : 4/10 < a < 7/10, 7/10 < a \leq 1\}$;
- and gf-$\Lambda$-closed sets = $\{C_a : 0 \leq a \leq 1\}$.

(1) $\Lambda$gf-closed $\nrightarrow$ fuzzy closed.

The $\lambda f$-open sets containing $C_{5/10}$ are $\{C_a : 6/10 \leq a \leq 7/10\}$. But $\text{cl}(C_{5/10}) = C_{6/10} \leq \{C_a : 6/10 \leq a \leq 7/10\}$. Thus $\text{cl}(C_{5/10}) \leq \lambda$ whenever $C_{5/10} \leq \lambda$.
and \( \lambda \) is \( \lambda \text{-f-open} \), which proves that \( C_{5/10} \) is \( \Lambda \text{-gf-closed} \). But \( C_{5/10} \) is not fuzzy closed.

(2) \( \text{gf-closed} \not\rightarrow \Lambda \text{-gf-closed} \).

For \( C_{7/10} \), \( C_1 \) is the only fuzzy open set containing \( C_{7/10} \) and hence \( C_{7/10} \) is \( \text{gf-closed} \). Since \( C_{7/10} \) is \( \lambda \text{-f-open} \) containing \( C_{7/10} \) we have \( C_{7/10} \subseteq C_{7/10} \). But \( \text{cl}(C_{7/10}) = C_1 \not\subseteq C_{7/10} \). Hence \( C_{7/10} \) is not \( \Lambda \text{-gf-closed} \). Thus \( C_{7/10} \) is \( \text{gf-closed} \) but not \( \Lambda \text{-gf-closed} \).

(3) \( \Lambda \text{-gf-closed} \not\rightarrow \lambda \text{-f-closed} \).

\( C_{5/10} \) is \( \Lambda \text{-gf-closed} \) and therefore \( \Lambda \text{-gf-closed} \). But \( C_{5/10} \) is not \( \lambda \text{-f-closed} \).

(4) \( \lambda \text{-f-closed} \not\rightarrow \Lambda \text{-gf-closed} \).

By (2) \( C_{7/10} \) is \( \text{gf-closed} \) and therefore \( \text{gf-\Lambda-closed} \). Now \( C_{7/10} \) is \( \lambda \text{-f-open} \) and \( C_{7/10} \subseteq C_{7/10} \). But \( \text{cl}_\lambda(C_{7/10}) = C_1 \not\subseteq C_{7/10} \) and hence \( C_{7/10} \) is not \( \Lambda \text{-gf-closed} \).

(5) \( \lambda \text{-f-closed} \not\rightarrow \text{fuzzy closed} \).

\( C_{3/10} \) is a \( \Lambda \)-fuzzy set and therefore it is \( \lambda \text{-f-closed} \), but \( C_{3/10} \) is not fuzzy closed.

(6) \( \Lambda \text{-gf-closed} \not\rightarrow \Lambda \text{-gf-closed} \).

By (5) \( C_{3/10} \) is \( \lambda \text{-f-closed} \) and hence \( \Lambda \text{-gf-closed} \). But \( C_{4/10} \) is fuzzy open and hence \( \lambda \text{-f-open} \) with \( C_{3/10} \subseteq C_{4/10} \) whereas \( \text{cl}(C_{3/10}) = C_{6/10} \not\subseteq C_{4/10} \) which proves that \( C_{3/10} \) is not \( \Lambda \text{-gf-closed} \).

(7) \( \text{gf-\Lambda-closed} \not\rightarrow \text{gf-closed} \).

\( \text{cl}_\lambda(C_{4/10}) = C_{4/10} \) and hence \( \text{cl}_\lambda(C_{4/10}) \subseteq \lambda \) whenever \( C_{4/10} \subseteq \lambda \) and \( \lambda \) is fuzzy open. Thus \( C_{4/10} \) is \( \text{gf-\Lambda-closed} \). But \( C_{4/10} \) is fuzzy open and \( C_{4/10} \subseteq C_{4/10} \) whereas \( \text{cl}(C_{4/10}) = C_{6/10} \not\subseteq C_{4/10} \). This proves that \( C_{4/10} \) is not \( \text{gf-closed} \).

**Remark 3.19.** The following concepts are independent. This fact can be seen from the Examples given.

(1) \( \Lambda \text{-gf-closedness} \) and \( \lambda \text{-f-closedness} \).

Example (1) : In Example 3.3, \( C_{3/10} \) is \( \lambda \text{-f-closed} \). But \( C_{3/10} \) is not \( \Lambda \text{-gf-closed} \) by (6) of Remark 3.18.
Example (2) : In Example 3.3, \( C_{5/10} \) is \( \Lambda \)-gf-closed by (1) of Remark 3.18. But \( C_{5/10} \) is not \( \lambda_f \)-closed.

(2) \( \Lambda \)-gf-closedness and gf-closedness.

Example (1) : In Example 3.3, \( C_{4/10} \) is \( \Lambda \)-gf-closed since \( \text{cl}_\lambda(C_{4/10}) = C_{4/10} \) by (7) of Remark 3.18. But it is not gf-closed by (7) of Remark 3.18.

Example (2) : In Example 3.3, \( C_{7/10} \) is gf-closed by (2) of Remark 3.18. But \( C_{7/10} \) is not \( \Lambda \)-gf-closed by (4) of Remark 3.18.

(3) \( \lambda_f \)-closedness and gf-closedness.

Example (1) : In Example 3.3, \( C_{3/10} \) is \( \lambda_f \)-closed by (5) of Remark 3.18. But \( C_{3/10} \) is not gf-closed for \( C_{4/10} \) is fuzzy open such that \( C_{3/10} \leq C_{4/10} \) whereas \( \text{cl}(C_{3/10}) = C_{6/10} \not\subseteq C_{4/10} \).

Example (2) : In Example 3.3, \( C_{7/10} \) is gf-closed by (2) of Remark 3.18. By (4) of Remark 3.18, \( C_{7/10} \) is not \( \Lambda \)-gf-closed and hence not \( \lambda_f \)-closed.

Remark 3.20. (1) Decomposition of a fuzzy closed set in terms of \( \lambda_f \)-closedness and gf-closedness.

By Theorem 3.12, a fuzzy set \( \mu \) is fuzzy closed \( \Leftrightarrow \mu \) is \( \lambda_f \)-closed and gf-closed.

By (3) of Remark 3.19, \( \lambda_f \)-closedness and gf-closedness are independent.

(2) Decomposition of a fuzzy closed set in terms of \( \lambda_f \)-closedness and \( \Lambda \)-gf-closedness.

Let \( \mu \) be fuzzy closed in \((X, \tau)\). By Proposition 3.16, \( \mu \) is \( \Lambda \)-gf-closed. Also by (2) of Remark 3.7, \( \mu \) is \( \lambda_f \)-closed. Hence \( \mu \) is \( \Lambda \)-gf-closed and \( \lambda_f \)-closed.

Conversely. Let \( \mu \) be \( \Lambda \)-gf-closed and \( \lambda_f \)-closed. By Proposition 3.16, \( \mu \) is gf-closed and hence by Theorem 3.12, \( \mu \) is fuzzy closed. Thus \( \mu \) is fuzzy closed \( \Leftrightarrow \mu \) is \( \Lambda \)-gf-closed and \( \lambda_f \)-closed. By (1) of Remark 3.19, \( \Lambda \)-gf-closedness and \( \lambda_f \)-closedness are independent.

Theorem 3.21. The union of two \( \Lambda \)-gf-closed sets is \( \Lambda \)-gf-closed.

Proof. Let \( A \) and \( B \) be any two \( \Lambda \)-gf-closed sets of a fuzzy topological space \((X, \tau)\). Let \( A \lor B \leq U \), where \( U \) is \( \lambda_f \)-open. Then \( A \leq U \) and \( B \leq U \). Since \( A \) and \( B \) are \( \Lambda \)-gf-closed,
cl(A) ≤ U and cl(B) ≤ U. Hence cl(A ∨ B) = cl(A) ∨ cl(B) ≤ U. Thus A ∨ B is Λgf-closed in X.

**Remark 3.22.** The intersection of two Λgf-closed sets need not be Λgf-closed as can be verified by the following Example.

**Example 3.23.** Let X = \{a, b\} and A : X → [0, 1] be defined as A(a) = 0.2, A(b) = 0.2. Then (X, τ) is a fuzzy topological space with τ = \{0_X, 1_X, A\}. The Λ-fuzzy sets in X are 0_X, 1_X and A. The fuzzy closed sets are 0_X, 1_X and A^1. Hence the Λf-closed sets are 0_X, 1_X, A and A^1. The λf-open sets are 0_X, 1_X, A and A^1 where A = \{(a, 0.2), (b, 0.2)\} and A^1 = \{(a, 0.8), (b, 0.8)\}. μ_1 = \{(a, 0.2), (b, 1)\} is Λgf-closed since 1_X is the only λf-open set in X containing μ_1. And μ_2 = \{(a, 1), (b, 0.2)\} is also Λgf-closed since 1_X is the only λf-open set in X containing μ_2. But μ_1 \wedge μ_2 = \{(a, 0.2), (b, 0.2)\} which is fuzzy open and hence λf-open, with \{(a, 0.2), (b, 0.2)\} ≤ \{(a, 0.2), (b, 0.2)\} whereas cl\{(a, 0.2), (b, 0.2)\} = \{(a, 0.8), (b, 0.8)\} ≤ \{(a, 0.2), (b, 0.2)\}. This verifies that μ_1 \wedge μ_2 is not Λgf-closed, in spite of μ_1 and μ_2 being Λgf-closed in X.

**Proposition 3.24.** If A is a Λgf-closed set of (X, τ) and A ≤ B ≤ cl(A), then B is a Λgf-closed set of (X, τ).

**Proof.** Let B ≤ U where U is λf-open in X. Since A ≤ B, A ≤ U. But A is Λgf-closed set in X, then cl(A) ≤ U. Also, B ≤ cl(A), cl(B) ≤ cl(A) ≤ U. Therefore B is Λgf-closed in X.

**Theorem 3.25.** If A is a λf-open and Λgf-closed set of (X, τ), then A is fuzzy closed in X.

**Proof.** Since A is λf-open and Λgf-closed, then cl(A) ≤ A and hence A is fuzzy closed in X.

**Theorem 3.26.** Let A be a Λgf-closed set in (X, τ).

1. If A is fuzzy regular open, then scl(A) and pint(A) are Λgf-closed sets.
(2) If $A$ is fuzzy regular closed, then $\text{pcl}(A)$ and $\text{sint}(A)$ are $\Lambda_{gf}$-closed sets.

Proof. (1) By the definitions, $\text{scl}(A) \geq A$ and $\text{pint}(A) \leq A$. But $A$ is fuzzy regular open, then $A = \text{int}(\text{cl}(A))$. Therefore $A$ is fuzzy semiclosed and fuzzy preopen. Thus $A$ is fuzzy semiclosed and hence $A = \text{scl}(A)$. Similarly we obtain $A = \text{pint}(A)$. Therefore $\text{scl}(A)$ and $\text{pint}(A)$ are $\Lambda_{gf}$-closed sets.

(2) By the definitions, $\text{pcl}(A) \geq A$ and $\text{sint}(A) \leq A$. But $A$ is fuzzy regular closed, then $A = \text{cl}(\text{int}(A))$. Therefore $A$ is fuzzy preclosed and fuzzy semiopen. Thus $A = \text{pcl}(A)$ and $A = \text{sint}(A)$. Therefore $\text{pcl}(A)$ and $\text{sint}(A)$ are $\Lambda_{gf}$-closed sets.

Theorem 3.27. Let $(X, \tau)$ be a fuzzy $T_{1/2}$ space. Then the following conditions are equivalent.

(1) $A$ is fuzzy closed.

(2) $A$ is $\Lambda_{gf}$-closed.

(3) $A$ is $gf$-closed.

Proof. Obvious.

Definition 3.28. A fuzzy set $A$ in $(X, \tau)$ is said to be $\Lambda_{gf}$-open in $(X, \tau)$ if and only if $A^1$ is $\Lambda_{gf}$-closed in $(X, \tau)$.

It is evident that every fuzzy open set of $(X, \tau)$ is $\Lambda_{gf}$-open in $(X, \tau)$ but not conversely.

In Example of Remark 3.18, $C_{5/10}$ is $\Lambda_{gf}$-open but not fuzzy open in $X$.

Theorem 3.29. The intersection of two $\Lambda_{gf}$-open sets in $(X, \tau)$ is $\Lambda_{gf}$-open.

Proof. This is obvious by Theorem 3.21.

The following Example shows that arbitrary union of $\Lambda_{gf}$-closed sets is not necessarily $\Lambda_{gf}$-closed.

Example 3.30. In Example 3.3, $\Lambda_{gf}$-closed sets $= \{C_a : 4/10 < a < 7/10 \text{ and } 7/10 < a \leq 1\}$ by Remark 3.18. $C_a$ is $\Lambda_{gf}$-closed for each $a$ such that $4/10 < a < 7/10$. 
and \( \vee C_a = C_b \) where \( b = \vee \{a \mid 4/10 < a < 7/10\} \). Hence \( \vee C_a = C_{7/10} \), where \( 4/10 < a < 7/10 \), which is not \( \Lambda gf \)-closed.

**Theorem 3.31.** A fuzzy set \( A \) is \( \Lambda gf \)-open in \((X, \tau)\) if and only if \( F \subseteq \text{int}(A) \) whenever \( F \) is \( \lambda f \)-closed in \((X, \tau)\) and \( F \subseteq A \).

**Proof.** Sufficiency. We first prove that \( A^1 \) is \( \Lambda gf \)-closed. Let \( A^1 \subseteq G \), where \( G \) is \( \lambda f \)-open. Hence \( G^1 \subseteq A \) and \( G^1 \) is \( \lambda f \)-closed. Then by the assumption \( G^1 \subseteq \text{int}(A) \) which implies that \( (\text{int}(A))^1 \subseteq G \), so \( \text{cl}(A^1) \subseteq G \). Hence \( A^1 \) is \( \Lambda gf \)-closed i.e., \( A \) is \( \Lambda gf \)-open.

Conversely. Let \( A \) be \( \Lambda gf \)-open. Then \( A^1 \) is \( \Lambda gf \)-closed and let \( F \) be a \( \lambda f \)-closed set contained in \( A \). Then \( A^1 \subseteq F^1 \). But \( A^1 \) is \( \Lambda gf \)-closed, then \( \text{cl}(A^1) \subseteq F^1 \). This implies that \( F \subseteq (\text{cl}(A^1))^1 = \text{int}(A) \). Thus \( F \subseteq \text{int}(A) \).

**Proposition 3.32.** If \( \text{int}(A) \subseteq B \subseteq A \) and \( A \) is \( \Lambda gf \)-open in \((X, \tau)\), then \( B \) is \( \Lambda gf \)-open in \((X, \tau)\).

**Proof.** Suppose \( \text{int}(A) \subseteq B \subseteq A \) and \( A \) is \( \Lambda gf \)-open in \((X, \tau)\). Then \( A^1 \subseteq B^1 \subseteq \text{cl}(A^1) \) and \( A^1 \) is \( \Lambda gf \)-closed. By Proposition 3.24, \( B^1 \) is \( \Lambda gf \)-closed in \((X, \tau)\) and hence \( B \) is \( \Lambda gf \)-open in \((X, \tau)\).

## 4. Fuzzy functions

We begin with the following notions:

**Definition 4.1.** A fuzzy function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be

(1) \( \lambda f \)-irresolute if \( f^{-1}(V) \) is \( \lambda f \)-open in \( X \) for every \( \lambda f \)-open set \( V \) of \( Y \).

(2) \( \lambda f \)-closed if \( f(F) \) is \( \lambda f \)-closed in \( Y \) for every \( \lambda f \)-closed set \( F \) of \( X \).

**Theorem 4.2.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be a \( \lambda f \)-irresolute fuzzy closed function. If \( A \) is \( \Lambda gf \)-closed in \( X \), then \( f(A) \) is \( \Lambda gf \)-closed in \( Y \).

**Proof.** Let \( A \) be a \( \Lambda gf \)-closed set of \( X \) and \( V \) be a \( \lambda f \)-open set of \( Y \) containing \( f(A) \). But \( f \) is \( \lambda f \)-irresolute, then \( f^{-1}(V) \) is \( \lambda f \)-open in \( X \) and \( A \subseteq f^{-1}(V) \). Since \( A \) is \( \Lambda gf \)-closed, hence \( \text{cl}(A) \subseteq f^{-1}(V) \) and \( f(A) \subseteq f(\text{cl}(A)) \subseteq V \). Since \( f \) is fuzzy closed, we obtain \( \text{cl}(f(A)) \subseteq V \). This shows that \( f(A) \) is \( \Lambda gf \)-closed in \( Y \).
Lemma 4.3. A function $f : (X, \tau) \to (Y, \sigma)$ is $\lambda f$-closed if and only if for each fuzzy subset $B$ of $Y$ and each $U \in \lambda \text{FO}(X)$ containing $f^{-1}(B)$, there exists $V \in \lambda \text{FO}(Y)$ such that $B \leq V$ and $f^{-1}(V) \leq U$.

Proof. Necessity. Suppose that $f$ is a $\lambda f$-closed function. Let $B \leq Y$ and $U \in \lambda \text{FO}(X)$ containing $f^{-1}(B)$. Put $V = (f(U^1))^1$. Then we obtain $V \in \lambda \text{FO}(Y)$, $B \leq V$ and $f^{-1}(V) \leq U$.

Sufficiency. Let $F$ be any $\lambda f$-closed set of $(X, \tau)$. Set $f(F) = B$, then $F \leq f^{-1}(B)$ and $f^{-1}(B^1) \leq F^1 \in \lambda \text{FO}(X)$. By hypothesis, there exists $V \in \lambda \text{FO}(Y)$ such that $B^1 \leq V$ and $f^{-1}(V) \leq F^1$. Therefore we obtain $V^1 \leq B = f(F) \leq V^1$. Hence $f(F) = V^1$ and $f(F)$ is $\lambda f$-closed in $(Y, \sigma)$. Therefore, $f$ is $\lambda f$-closed.

Theorem 4.4. Let $f : (X, \tau) \to (Y, \sigma)$ be a fuzzy continuous $\lambda f$-closed function. If $B$ is a $\Lambda gf$-closed set of $(Y, \sigma)$, then $f^{-1}(B)$ is $\Lambda gf$-closed in $(X, \tau)$.

Proof. Let $B$ be a $\Lambda gf$-closed in $(Y, \sigma)$ and $U$ be a $\lambda f$-open set of $(X, \tau)$ containing $f^{-1}(B)$. Since $f$ is $\lambda f$-closed, then by Lemma 4.3 there exists a $\lambda f$-open set $V$ of $(Y, \sigma)$ such that $B \leq V$ and $f^{-1}(V) \leq U$. But $B$ is $\Lambda gf$-closed in $(Y, \sigma)$, then $\text{cl}(B) \leq V$ and hence $f^{-1}(B) \leq f^{-1}(\text{cl}(B)) \leq f^{-1}(V) \leq U$. Since $f$ is fuzzy continuous, $f^{-1}(\text{cl}(B))$ is fuzzy closed and hence $\text{cl}(f^{-1}(B)) \leq U$. This shows that $f^{-1}(B)$ is $\Lambda gf$-closed in $(X, \tau)$.

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References


(1,2) Mathematics Section, Faculty of Engineering and Technology, Annamalai University, Chidambaram, Tamil Nadu, India.

E-mail address, (1): k_g_balu@yahoo.co.in
E-mail address, (2): ssm_3096@yahoo.co.in

(3) Department of Mathematics, P. M. Thevar College, Usilampatti, Madurai Dt., Tamil Nadu, India.

E-mail address: siingam@yahoo.com